Structural Treatment of Time-Varying Dynamical System Networks in the Light of Hybrid Symmetries

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«*Mein Leben?!:* ist kein Kontinuum! (nicht bloß durch Tag und Nacht in weiß und schwarze Stücke zerbrochen! Denn auch am Tage ist bei mir der ein Anderer, der zur Bahn geht; im Amt sitzt; büchert; durch Haine stelzt; [...] schreibt; Tausendsdenker; auseinanderfallender Fächer; [...]: ein Tablett voll glitzernder snapshots.

Kein Kontinuum, kein Kontinuum!: so rennt mein Leben, so die Erinnerungen (wie ein Zuckender ein Nachtgewitter sieht): [...].

Aber als majestätisch fließendes Band kann ich mein Leben nicht fühlen; nicht ich! (Begründung).»

Arno Schmidt: *Aus dem Leben eines Fauns.*

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Abstract

Numerous dynamical systems describing real world phenomena exhibit a characteristic fine structure which means that they are composed of smaller subsystems interacting with each other. This interaction structure can be represented by a coupling network whereby the original system can be viewed as a network of dynamical systems and is commonly termed *coupled cell system*. Since reality crucially depends on time, derived models generally tend to be subject to temporal changes as well. Particularly in applications involving technology, this temporal evolution often occurs as a consequence of an instantaneously varying network structure; communication networks provide a prominent class of examples.

In this thesis, time-varying dynamical system networks are analyzed on the grounds of the following two structural aspects: Firstly, instantaneous modifications of the underlying coupling network generally lead to non-smooth vector fields and, secondly, a fixed network structure naturally introduces symmetries to the according system. These analytical and algebraic observations trigger the system's description as a hybrid dynamical system with local symmetry information. In search of global structure for systems of such kind, a global symmetry framework for hybrid dynamical systems formulated in terms of hybrid automata is unfolded that takes into account both discrete transition graph symmetries and local dynamical systems' symmetries giving rise to the concept of *hybrid symmetries*. Moreover, hybrid periodicity and hybrid spatio-temporal symmetries are discussed – all in the light of classical symmetries in the context of dynamical systems.

Restricted to a special class of switched systems which induce hybrid automata by the choice of a switching signal, symmetry-induced switching strategies

termed *orbital* switching signals are investigated and stability issues of switched linear systems are addressed. Against this theoretical background, examples of time-varying dynamical system networks are treated both structurally and numerically for orbitally switched coupling networks. With a view to applications, orbital switching can be interpreted in terms of cyclically evolving network perturbations.

Zusammenfassung

Zahlreiche dynamische Systeme, die der Beschreibung realer Phänomene dienen, weisen eine charakteristische Feinstruktur auf, d. h. sie setzen sich aus kleineren Systemen zusammen, die sich wechselseitig beeinflussen. Die Struktur dieses Zusammenspiels kann als Kopplungsnetzwerk dargestellt werden, womit das ursprüngliche System als ein Netzwerk dynamischer Systeme betrachtet werden kann, das in der englischsprachigen Literatur üblicherweise als *Coupled Cell System* bezeichnet wird. Da die Realität selbst in höchstem Maße zeitabhängig ist, unterliegen auch zu ihrer Beschreibung entwickelte dynamische Systeme prinzipiell zeitlichen Veränderungen. Insbesondere in technologischen Anwendungen rührt diese Zeitabhängigkeit oftmals von einer sich instantan verändernden Netzwerkstruktur her; dieses ist beispielsweise bei Kommunikationsnetzwerken häufig der Fall.

Der Schwerpunkt dieser Arbeit liegt in der Analyse solcher zeitabhängigen Netzwerke dynamischer Systeme, die vorrangig auf den folgenden beiden strukturellen Aspekten basiert: Zum einen führen die instantanen Modifikationen des zugrundeliegenden Kopplungsnetzwerkes im allgemeinen auf nichtglatte Vektorfelder und zum anderen induziert eine feste Netzwerkstruktur auf natürliche Weise Symmetrien des entsprechenden dynamischen Systems. Diese analytischen und algebraischen Fakten veranlassen die Beschreibung des Systems als ein hybrides dynamisches System mit lokaler Symmetrie-Information. Dies motiviert die Entwicklung eines globalen Symmetriekonzeptes für hybride dynamische Systeme in Form hybrider Automaten, welches sowohl die diskreten Symmetrien des Transitionsgraphen wie auch die klassischen Symmetrien der lokalen dynamischen Systeme berücksichtigt. Im Zuge dieser Konstruktion erfolgt die Definition *hybrider Symmetrien* sowie deren algebraische und die hybride Dynamik betreffende Behandlung. Darüberhinaus werden hy-

bride Periodizität und in diesem Zusammenhang hybride raum-zeitliche Symmetrien im Hinblick auf klassische Symmetrien diskutiert.

Auf der Grundlage dieser Betrachtungen werden durch Symmetrien generierte Schaltstrategien – sogenannte *orbitale* Schaltsignale – für eine spezielle Klasse von *Switched Systems* behandelt, welche als verallgemeinerte hybride Automaten verstanden werden können. In diesem Kontext werden Stabilitätsfragen spezieller hybrider Systeme untersucht. Vor diesem theoretischen Hintergrund schließt sich die strukturelle und numerische Analyse zeitabhängiger Systemnetzwerke an, die durch das orbitale Umschalten ihrer Kopplungsnetzwerke charakterisiert sind. Im Hinblick auf Anwendungen kann diese Art von orbitalem Schalten als zyklisch wandernde Netzwerkstörung interpretiert werden.

Contents

Chapter 1

Introduction

This thesis is motivated by dynamical phenomena of networks of dynamical systems that change in time. It approaches the issue in terms of hybrid systems and sets up a notion of hybrid symmetry which is linked to special evolution patterns of dynamical system networks. Before we go into detail, we embed the subject into its scientific surroundings, discuss the motivating ideas and introduce the fundamental components that will be involved in the analysis. The structural composition of the thesis is pointed out afterwards.

Concerning the Mathematical Analysis of the World

Reality may be viewed as a highly dynamic multilayered phenomenon of countless interacting instances. These instances may be understood as units characterized by spirit and purpose or simply as *systems*. The mathematical discipline that is set up for handling the temporal development of a system's states is the theory of *dynamical systems*. Since – on each scale of observation – reality tends to be extremely intricate and complex, one cannot severly pursue the plan of formally understanding the world as a whole. Naturally, the interest focuses on comparably small subsystems which to comprehend drives the researchers' efforts.

Reality is tightly woven of interdependencies. When trying to perceive aspects of the real world from a scientific point of view, one sees network structures shine through on each scale the eye accomodates to. This applies to every aspect of life, be it nature, technology, economy or social relationship: The world is interspersed with *networks*.

Besides, reality is non-static in every respect. What makes things even more difficult and considerably harder to trace, is the awareness that as a matter of principle everything that surrounds us crucially depends on *time*. Especially not only a system's state but the system *itself* is subject to permanent temporal changes.

Consequently, we have to face the challenge of describing and analyzing reality in terms of *time-varying networks of dynamical systems* if we aim to obtain a reasonable mathematically formalized picture detail of what is commonly called *the real world*. However, there is no reason to blindly assume that networks vary smoothly in time. Already the consideration of mobile ad-hoc communication networks or interacting robots strongly suggests that there are quite a number of distinguished cases of dynamical system networks where the spatially discrete network structure evolves non-smoothly in time. Even the wink of the observer's eye or the quantization of energy keeps us away from a world that can in principle be experienced in a smooth manner. It is this line of thought which introduces mixed discrete-continuous traits to the issue and leads into the framework of *hybrid systems* which are characterized by a combination of discrete events and continuous flow dynamics.

Once the awareness of network structures is established, the strongly and indeed naturally related notion of *symmetry* enters the discussion, since the automorphisms of a graph typify a geometrically very descriptive occurence of symmetry and symmetries constitute an important form of additional structure from the dynamical system point of view inasmuch as they deeply influence the dynamical behaviour of the system under examination. In consideration of an evolving network structure, symmetries themselves are bound to change and in view of the system's hybrid formulation a newly discovered *hybrid symmetry* structure is given rise to which turns out to be closely connected to special network evolution patterns.

This work studies the relation between hybrid symmetries of a hybrid system and the structural as well as dynamical properties of the according hybrid dynamics driven by symmetry-induced switching strategies. The results are illustrated by means of time-varying dynamical system networks as prototypical examples of hybrid systems possessing non-trivial hybrid symmetries. In order to give a more precise description of the essential contents and achievements of this thesis, we turn towards a more elaborate characterization of the single building blocks involved and point out the ideas as well as their realization in more detail.

Dynamical Systems

The concept of *dynamical systems* represents an essential instance in order to model various kinds of dynamic processes which appear in divers aspects of human life, such as nature, technology and economics. Broadly speaking, the notion of a dynamical system provides a mathematically formalized model capturing the temporal development of a system's state. From a strict mathematical point of view, the most natural, but actually most abstract description of dynamical systems is rooted in the field of automorphism semigroups – or sets equipped with semigroup actions, to put it differently. More concretely, *time*, which is modeled as a discrete or continuous semi-group, acts on the *phase space* of the system under consideration. It is this action which specifies the *time-evolution law* (or more simply, the *flow* of the system) that encodes the passage in time of single states within their ambient space.[1] To be more precise, an abstract dynamical system Ψ on a non-empty set Ω – the *state* or *phase space* – is a triple (Ω, G, Φ) further consisting of a semi-group G and an action $\Phi : G \times \Omega \to \Omega$ of G on Ω, i. e. the identities

$$\Phi(e, \omega) = \omega \quad \text{and} \quad \Phi\left(g, \Phi(h, \omega)\right) = \Phi(gh, \omega)$$

hold for all $g, h \in G$ and $\omega \in \Omega$ with $e \in G$ denoting the neutral element.

On a very general level, given a dynamical system Ψ, the main interest focuses on the system's *trajectories* – which are the semi-group orbits $G\omega_0 = \{\Phi(g, \omega_0) \mid g \in G\} \subset \Omega$ of an initial state $\omega_0 \in \Omega$ – in order to obtain a sufficient understanding of the dynamics. Hence, trajectories are the paths of single states in phase space. However, for different reasons, it is not sufficient or even impossible in most cases to understand a dynamical system solely in terms of

[1] One should stress that commonly using the term *dynamical system* this evolution rule is *deterministic*, i.e. given the current state of the dynamical system, there is a unique future state for each later point in time.

its individual trajectories. Thus, appropriate methods and tools for the local as well as global analysis of dynamical systems are necessary. These include – amongst others – notions of stability and bifurcation theory of dynamical systems.

Notably, in case of continuous state and time, a dynamical system straightforwardly gives rise to an *ordinary differential equation*: For $X \subset \mathbb{R}^n$ and $G = (\mathbb{T}, +)$, where $\mathbb{T}$ is one of the (semi-)groups $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$, a dynamical system is a continuously differentiable map $\Phi : \mathbb{T} \times X \to X$ such that the induced one-parameter family of maps $\{\phi_t\}_{t \in \mathbb{T}}$ with

$$\phi_t = \Phi(t, \cdot) : X \to X,\ x \mapsto \Phi(t, x)$$

satisfies the simple, but natural relations

$$\phi_0 = \mathrm{id}_X \qquad \text{and} \qquad \phi_t \circ \phi_s = \phi_{t+s}$$

for all $s, t \in \mathbb{T}$. The definition of a vector field $f : X \to \mathbb{R}^n$ by mapping a state $x \in X$ to the tangent vector of the curve $\phi^x : \Phi(\cdot, x) : \mathbb{R} \to X$, $t \mapsto \phi_t(x)$ at $t = 0$, i. e.

$$f(x) = \frac{d}{dt}\phi_t(x)\bigg|_{t=0},$$

leads to the ordinary differential equation

$$\dot{x}(t) = f(x(t))$$

with $x(t) = \phi_t(x)$. Conversely, under suitable conditions on the vector field, an ordinary differential equation induces – at least locally – a dynamical system. In case, this is also globally possible, one does not have to distinguish between a dynamical system as an $\mathbb{R}$-action and the corresponding differential equation. As a classic introductory textbook on dynamical systems, [HS74] should be mentioned primarily, and for a more recent and comprehensive compendium, one should confer [KH98].

In this work, the focus concentrates on the *structural* treatment of dynamical systems which is why we decide on rather plain requirements with respect to the occuring dynamical systems: We assume $X = \mathbb{R}^n$ and $\mathbb{T} = \mathbb{R}$ and intentionally avoid to linguistically make a difference between dynamical systems and ordinary differential equations.

Now, we address the difference between *autonomous* and *non-autonomous* dynamical systems, which also plays a specific role in the upcoming considerations. A dynamical system $\dot{x} = f(x)$ is said to be *autonomous* if the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ does not explicitely depend on time. In this case, the interpretation is that the time-evolution law is fixed and does not change in time. Under mild assumptions, locally, the initial value problem

$$\dot{x} = f(x), \quad x(t_0) = x_0,$$

for some initial value $x_0 \in X$ has a unique solution $x(t)$ which is a curve $I_{x_0} \to X$ defined on an interval $I_{x_0} \subset \mathbb{R}$ that satisfies the differential equation, namely $\dot{x}(t) = f(x(t))$ for all $t \in I_{x_0}$.

If explicit time-dependence is on hand, i. e. the equation has the form $\dot{x} = f(t, x)$ with vector field $f : \mathbb{T} \times X \to X$, we speak of a *non*-autonomous dynamical system and imagine a time-evolution law varying temporally as the state itself evolves. What is special about non-autonomous dynamical systems is that they have the *cocycle property*: Let the initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$ have a unique solution $x(x_0, t_0, t)$ and write $\phi^t_{t_0}(x_0) = x(x_0, t_0, t)$; then the two-parameter family $\{\phi^t_s\}_{s,t\in\mathbb{R},s\leq t}$ of maps fulfills

$$\phi^t_t = \mathrm{id}_X \qquad \text{and} \qquad \phi^t_s \circ \phi^s_r = \phi^t_r$$

for all $r, s, t \in \mathbb{R}$. In many cases, explicit time-dependence results from time-variant external influences and generally introduces additional complications with respect to analysis. If there is periodicity in time meaning $f(t + T, x) = f(t, x)$ for all $t \in \mathbb{R}$, $x \in X$ and some period $T \in \mathbb{R}$, interesting symmetry properties may arise (so-called *spatio-temporal symmetries*, cf. [Fie88] or [GS02], for instance); we turn to this topic later on in Chapter 6 .

Symmetries, Equivariant Dynamical Systems & Coupled Cell Systems

A dynamical system which exhibits additional structure in terms of *symmetries* is said to be *equivariant*. In this framework, the term ≪symmetry≫ names a transformation of the phase space which preserves the system's dynamic behaviour by taking trajectories to trajectories. The natural language for the description of global symmetry properties is that of *groups* since (global) symmetries always form the algebraic structure of a group acting on the phase space. This translates to an *equivariance* condition on the vector fields determining the induced differential equations. Symmetries of dynamical systems may arise from the system geometry (e.g. experimental set-ups in physics or natural arrangements and interdependencies in biology), from simplifying modeling assumptions or from the idealization of real world phenomena (e.g. the spherical symmetry of the Earth).

Intuitively, it appears to be obvious that the presence of symmetries strongly shapes the overall dynamical behaviour and, generally, offers the possibility of reducing its complexity. Furthermore, symmetries generate *flow-invariant subspaces* of the phase space: These are regions that – once entered – cannot be left again, and for special purposes it is convenient to restrict the system to such a subspace. What motivates an extensive treatment of equivariant dynamical systems, furthermore, is the occurence of certain dynamical phenomena which are typically not observed in generic systems lacking symmetries, for instance the existence of heteroclinic cycles. An exceedingly important branch of equivariant dynamical system analysis is *equivariant bifurcation theory*. Here, in presence of symmetries, the qualitative change of dynamics is studied while a parameter is varied: At bifurcation points stability properties of solutions are affected, the orbit type may change and the symmetry of trajectories may be destroyed (*symmetry breaking*). Equivariant dynamical systems are treated in the textbooks [GSS88] and [GS02], for instance.

The presence of network structures provides a rich breeding ground for symmetries. A *coupled cell system* is a network of dynamical systems – referred to as *cells* – which are coupled together according to an underlying coupling network – the *coupled cell network* – and thus influence each other dynamically. The concept of coupled cell systems both strictly formalizes and considerably extends the theory of equivariant dynamical systems which are composed of smaller subsystems and whose symmetries stem from their coupling architecture.

Global network symmetries are known to influence the overall dynamical behavior strongly causing phenomena like synchrony of subsystems (patterns of synchronized cells) or phase relations, for instance. However, as one can observe, this global kind of symmetry is not the only particular structure imposed on a dynamical system network to create such dynamical properties. Furthermore, global symmetries are highly sensitive with regard to perturbations concerning the network topology describing the coupling. In order to deal with problems of that type, the formalism of coupled cell systems provides a more general notion of symmetry which is of *local* nature, in contrast to classical global symmetries. As a consequence, the algebraic object gathering all the symmetry information of a system – a group in the case of classically equivariant systems – is replaced by a more complex and robust object: the *symmetry groupoid*.

The first main results in this field of study (stated by Golubitsky, Stewart and co-workers in [GS06], for instance) comprise the complete combinatorial classification of *robust* synchrony patterns (*balanced equivalence relations*) and the reduction of coupled cell systems with regard to a chosen pattern of synchrony (*quotient networks*). The preceding term «robust» refers to another major feature of coupled cell systems: The results stated above apply to all vector fields which are compatible with the underlying coupled cell network. In particular, they do not depend on the explicit form of the vector fields, but solely on the network structure given by the coupling. These characteristics and the attribute of being composed of more elementary subsystems make the idea of coupled cell systems amenable to applications with regard to modeling and analysis issues of real world problems. An introduction to the formalism of coupled cell systems can be found in [SGP03] while [GS06] should be consulted as a survey comprising the state of research until 2006.

Hybrid Dynamical Systems

In the broadest sense, *hybrid systems* are dynamical systems which involve the interaction of qualitatively different types of dynamics. More specifically, the idea of *hybrid dynamical systems* considered here is born out of the combination of *continuous* dynamical behavior (*flowing*, described by a continuous state dynamical system) and *discrete* event dynamics (*jumping*, given by a finite state system). Since discrete dynamics may affect the continuous evolution of the system (and vice versa), the analysis and design of hybrid systems tends to be substantially more difficult and complex than in the purely discrete or continuous case.

The notion of hybrid dynamics provides an appropriate framework for system modelling in a wide range of engineering applications: e.g. mechanical systems (collisions), electrical circuits (hybridly behaving diodes and transistors, charging of capacitors interrupted by switching, see e. g. [HCS01]), chemical process control (control of chemical reactions by valves and pumps, see [MRB+07] for the modeling and the dynamics of a class of controlled reverse flow reactors), air traffic management, scheduling of automated railway systems and embedded computation. A possibility to formalize hybrid systems and set up a mathematical environment for their modeling and analysis is to make use of *hybrid automata* which – in outlines – are dynamical systems networked by a transition graph provided with rules supervising the transient behavior.

Specifying initial and final states, additionally, a hybrid automaton becomes a *state transition system* which yields the possibility to consider reachability questions algorithmically. For hybrid systems this translates to *safety properties*, for instance, which are of vital importance whenever safety of real world systems is concerned; automated highway systems or flight control are only two noteworthy instances.

From the dynamical systems' point of view, even very simply structured hybrid systems may exhibit exotic, occasionally unwanted dynamical behavior (the *Zeno* property, cf. particularly [ZJLS01], or *blocking*). Therefore, analysis requires different approaches and new non-standard techniques. One of the extremely rare introductory textbooks on general hybrid systems is [vdSS00]. The hybrid automata setting and first dynamical considerations can be found in [LJS+03].

Time-Varying Dynamical System Networks and Symmetries - The Root Idea

From a structural point of view, a dynamical system network is a significantly richer object than just an ensemble of dynamical systems each being on its own. It is the network topology which imposes a communication grid between the systems joining them together and letting them interact to provoke collaborative dynamical behaviour. Therefore, what is special about such networked systems is the interlocking of two qualitatively *completely different* types of structure: The spatially discrete network structure on the one hand and the continuous dynamics of the single systems on the other hand and both tightly connected to form a unit.

In that spirit, a dynamical system network (or *coupled cell system*) can naturally be viewed as a dynamical system itself defined on a *high-dimensional* phase space, namely the Cartesian product of all individual phase spaces involved. Thus, if this global dynamical system experiences temporal development meaning that it is non-autonomous, this may – as a matter of fact – be a consequence of basically two substantially distinct factors due to its peculiar structure. Firstly, one of the single systems (or a group or even all of them) may explicitely depend on time in a smooth manner while the network architecture is temporally fixed. In this situation, we come upon a classical non-autonomous dynamical system $\dot{x} = F(t, x)$. Secondly, the underlying network topology may be non-static and undergo *instantaneous* changes in the course of time.[2] Certainly, both kinds of time-dependence may occur simultaneously, but this thesis strictly focuses on the latter case. Non-autonomous dynamical systems of that kind may arise from the discretization of partial differential equations with evolving boundary conditions, for instance.

The crucial effect of the instantaneous network evolution is the general loss of smoothness or even continuity with respect to the vector field. All that remains

[2] Of course, there are two further mechanisms to give rise to explicit dependence on time: the instantaneous structural change of a subsystem and the smooth variation of the underlying network structure. Both types of modification, however, are not in line with the according object's categorical origin and are therefore not considered here.

is the former regularity between two time instants of network modification, i. e. during the time the network rests. It is this observation which imperatively suggests that we have left the classical stage of dynamical systems and already perform in the regime of *hybrid* systems.

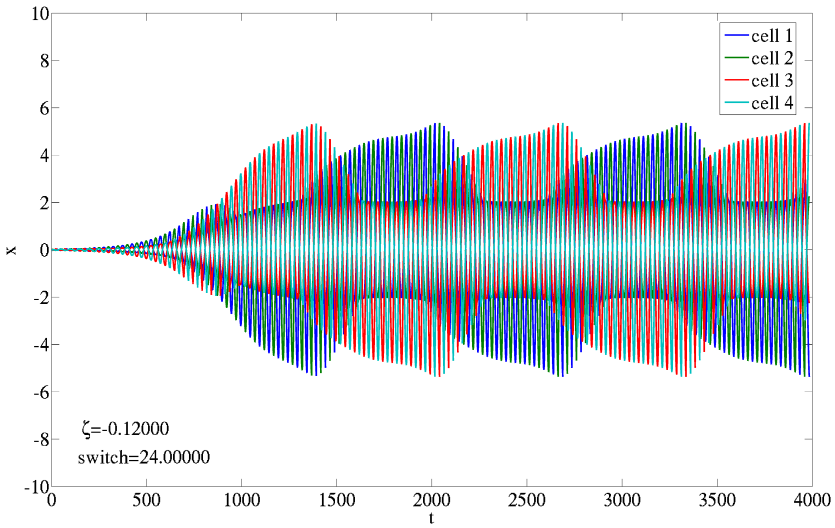


FIGURE 1.1: Numerical example: Plot of a solution of a time-varying dynamical system network with two distinct network structures involved; each dynamical system involved is 8-dimensional and the coordinates x_1, x_3, x_5 and x_7 are plotted.

Concerning network dynamics, we assume that the number of nodes is constant meaning that as time goes by nodes can neither vanish nor be created, changes are completely induced by the variation of edges. Under these circumstances, there are only finitely many ways for a given finite network to change. It appears reasonable to administrate all possible dynamical systems resulting from different network structures as well as the admissible transitions from one coupling state to another: It may be forbidden that all couplings vanish at once, for instance, or that some sequence of configurations is impossible. Again, this approach formally brings forth a network of dynamical systems, but notably the underlying graph structure bears a different meaning compared to a coupled cell system. While the edges are interpreted as spatial couplings between the subsystems in the case of a coupled cell system, they express the temporal evolution of the varying system in our recently considered case. As soon as the edges are provided with conditions that manage the passage from one node to another along an edge, the original time-varying dynamical system network has been transformed to a *hybrid automaton*, a specific way of describing a hybrid system based on a transition graph.

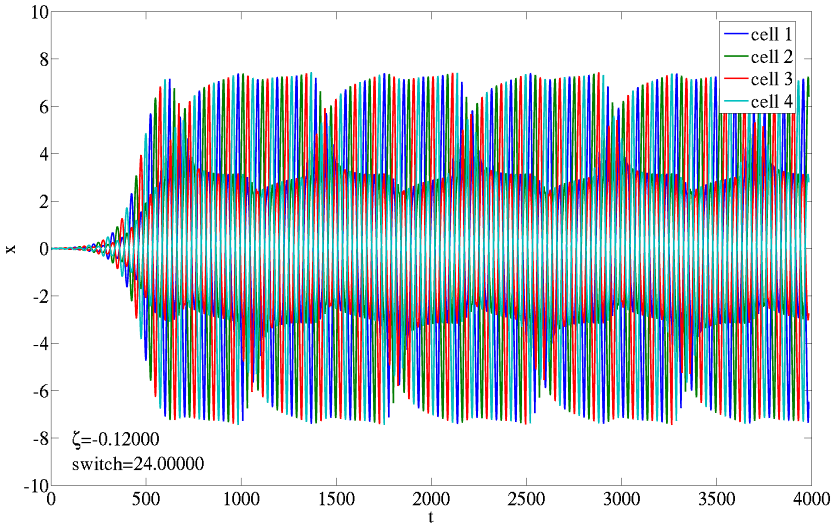


FIGURE 1.2: Dynamics of another time-varying dynamical system network with four possible network structures involved

The substantial effect of putting a time-varying dynamical system network into the framework of hybrid automata from a superior point of view consists in the *autonomization* of the original mathematical object, i. e. the former non-autonomous system becomes an *autonomous* object in the category of hybrid systems. Now, as hinted at before, network structures generate a special form of dynamical system symmetries strongly connected to the automorphisms of the according coupling graph.

Examples as illustrated in Figures 1.1 and 1.2 suggest the presence of some kind of symmetries that appear to be connected to the way the network changes. The dynamical systems revealing the dynamics displayed in Figures 1.1 and 1.2 are treated throughout the thesis, in particuliar in Chapters 3, 5 and 6.

What is an apparently incoherent miscellany of time-dependent symmetries that lacks a unifying or at least connective description in algebraic terms, by autonomization becomes an expectant collection of coexistent elements demanding an algebraic arrangement of the overall symmetry information genetically contained in the derived hybrid system. This is the point where *hybrid symmetries* are born. In order to analyze a possible interrelation between the patterns which the dynamics of a time-varying network of dynamical systems exhibit with its symmetries, the development of a global symmetry framework for hybrid dynamical systems is necessary.

Subject and Structure of this Thesis

This thesis is concerned with the structural analysis of time-varying dynamical systems in the setting of hybrid dynamical systems with its main focus on symmetries. It establishes a global symmetry framework for hybrid dynamical systems and discusses in detail a special class of switching signals which are induced by hybrid symmetries. In this context, stabilization issues for linear hybrid systems are addressed. Switched systems with respect to symmetry-induced signals are found to adequately describe time-varying dynamical system networks with perturbations or link failures moving periodically through the network.

The thesis is structured as follows. In **Chapter 2**, the concept and formalism of hybrid dynamical systems is introduced starting with the formal definition of *hybrid automata* before discussing *hybrid dynamics*. Widely based on recent literature, the exposition is characterized by an own accent originating from the intention to find a closed form for presenting the hybrid system framework. Most significantly, in the course of these modifications, purely time-dependent switching is integrated into the hybrid automaton framework.

Chapter 3 carefully documents the development of a symmetry setting for hybrid dynamical systems and examines hybrid dynamics in presence of such *hybrid symmetries*. At the outset of Chapter 3, the nature of symmetries for classical dynamical systems is sketched in order to provide the reader with the basic concepts that are to be generalized to the hybrid case afterwards. The construction of hybrid symmetries is carried out in two major steps: Firstly, abstractly coupled equivariant dynamical systems – termed *dynamical $\mathscr{T}$-systems* based on a directed graph $\mathscr{T}$ – are considered. These are dynamical system networks whose edges' meaning is undetermined. For systems of that type, a notion of symmetry is created and studied in terms of single dynamical systems and with a view to algebraic properties. In this process a new equivariance-like property arises for the according vector fields – termed *weak equivariance* – which may be interpreted as a kind of *spatio-spatial symmetry* in the style of *spatio-temporal symmetries.* Truly, this type of equivariance appears as a generalization of classical equivariance for vector field families set in relationship via a graph. The second step realizing the passage from dynamical $\mathscr{T}$-systems to hybrid dynamical systems consists in the inclusion of instances supervising the switching between adjacent dynamical systems. This leads to the completion of *hybrid symmetries* that – according to the qualitatively heterogeneous, composite structure of hybrid systems – originate from the matched interlocking of discrete and continuous symmetries which is also discussed in detail. Subsequently, the immediate consequences for hybrid dynamics are addressed coming along with the presence of hybrid symmetries. The chapter closes with the examination of *hybrid fixed-point spaces* which turn out to exhibit considerably weaker properties compared to their classical flow-invariant counterparts. This awareness constitutes a major difference between hybrid and classical symmetries arising due to the structural otherness of hybrid systems.

After Chapter 3 has covered the hybrid analog of (purely spatial) symmetries, **Chapter 4** holds the treatment of *hybrid spatio-temporal symmetries.* To this end, periodicity in the dynamics of hybrid automata is inspected and characterized structurally. Based thereupon, the hybrid spatio-temporal symmetries of a single execution are established and characterized. The observation that an execution of a hybrid automaton induces a hybrid automaton related to the original one, gives rise to the question concerning the connection between these systems with regard to hybrid symmetries; in the case of hybrid spatio-temporal symmetries, it is shown that this symmetry information is passed on to the induced systems. For the purpose of gaining a deeper understanding of the impact of hybrid spatio-temporal symmetries, the perspective is slightly changed to hybrid automata with prescribed periodic discrete state maps. Analogous to the classical case, the hybrid time-T map with T denoting the period of the underlying switching signal decomposes to an iterate of a related map which is connected to the hybrid spatio-temporal symmetries of the switched system. This result allows for a characterization of spatio-temporally symmetric executions as fixed-points of certain maps.

In **Chapter 5**, the focus completely turns to *switched systems* whose discussion serves the purpose of examining a hybrid dynamical system from another point of view bringing the role of switching signals to the fore and the concept of hybrid symmetries is briefly adapted to switched systems. Afterwards, a special class of switching signals generated by hybrid symmetries and termed *orbital switching* is centered. In particular, the effect of conjugation is studied for orbital switching signals. The role of a signal's *switching time* with respect to symmetries is figured out and orbital switching is shown to give rise to hybrid spatio-temporal symmetries. Thus it imposes a particular structure on the hybrid return map with respect to the period of the orbital signal. As an application of this symmetry-based decomposition, the stabilization of switched linear systems is investigated.

It is then that **Chapter 6** turns towards time-varying networks of dynamical systems against the background of the theoretical results developed in Chapters 3 to 5. Beginning with the formal introduction of *coupled cell systems* with fixed coupling topology, a special class of globally symmetric coupled cell systems is described and exposed to smoothly varying and periodically spreading network perturbations. In the course of a discretization process, a hybrid automaton model is derived for this time-varying dynamical system network and its hybrid symmetry properties are revealed. Numerical experiments illustrate symmetry-shaped dynamics and stabilization phenomena in line with the results of the main part of this thesis.

Finally, **Chapter 7** provides a detailed conclusion of this work. After a retrospective summary, the most important findings are pointed out followed by an outlook on possible future research concerning this topic.

Chapter 2

Hybrid Dynamical Systems

This chapter serves as an introduction to the mathematical description of hybrid dynamical systems and is composed as follows: Firstly, Section 2.1 provides and discusses a formal definition of hybrid dynamical systems this thesis fundamentally builds on. Thereupon Section 2.2 deals with the depiction of the dynamics such systems give rise to.

2.1 The Concept of Hybrid Dynamical Systems

We mark the beginning of seeing and understanding things from a hybrid point of view by the rigorous definition of *hybrid dynamical systems*, the primary concept this thesis builds on. Having laid this formal grounding, we inspect and interpret these exceptional mathematical objects relating them to what is classically known as *dynamical systems* and thereby discovering their dynamical meaning and importance.

The following definition is almost identical to the definitions given in [LJS$^+$03] and [SJLS05] except for the concept of *clocks* which I have incorporated in order to include mixed state-time and pure time-dependent switching.[1] An even more abstract and thus more general definition can be found in [vdSS00] whose

[1] To avoid misconception, I would like to explicitely point out that – of course – time-dependent switching is an important mechanism and has been extensively treated for a couple of years now, but it has not been considered as a specific type of switching for general hybrid automata.

degree of generality does not meet my current intentions, however. Please note that in the literature the term *hybrid automaton* is much more common than *hybrid dynamical system* (at least with a view to Definition 2.1.1) which I prefer to use throughout this thesis; the reason for my choice is that my interest severely focuses on the dynamical understanding of hybrid automata, i. e., the nature of its dynamics, and the term *hybrid automata* unnecessarily conceals the dynamical aspect and so does not appear suitable to me.

Given a directed graph $\mathscr{T} = (\Lambda, \mathcal{E})$ with vertices Λ and edges $\mathcal{E} \subset \Lambda \times \Lambda$, we denote by $\mathfrak{s}$ and $\mathfrak{t}$ the *source* and *tail* maps

$$\mathfrak{s} : \mathcal{E} \to \Lambda, \quad \mathfrak{s}(i,j) = i \quad \text{and} \quad \mathfrak{t} : \mathcal{E} \to \Lambda, \quad \mathfrak{t}(i,j) = j. \tag{2.1}$$

These maps coincide with the projections onto the first and second factor of the product $\Lambda \times \Lambda$.

2.1.1 Definition (Hybrid Dynamical System). Let $\mathbb{T}$ denote continuous time, i. e. $\mathbb{T} \subset \mathbb{R}$. A *hybrid dynamical system* (or *hybrid automaton*) $\mathscr{H}$ of dimension n is a septuple $\mathscr{H} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{C}, \mathcal{G}, \mathcal{R})$ composed of the following data:

- a (countable) set Λ of *discrete states*,
- a collection $\mathcal{E} \subset \Lambda \times \Lambda$ of *discrete transitions*,
- a Λ-indexed family of *domains* $\mathcal{D} = \{D_\lambda\}_{\lambda \in \Lambda}$ with $D_\lambda \subset \{\lambda\} \times \mathbb{R}^n$,
- a *vector field family* $\mathcal{F} = \{f_\lambda : D_\lambda \to \mathbb{R}^n\}_{\lambda \in \Lambda}$,
- a collection $\mathcal{C} = \{T_\lambda\}_{\lambda \in \Lambda}$ of *clocks* where a (continuous) clock is a one-dimensional dynamical system on $\mathbb{T}$ with vector field $T_\lambda \equiv 1$,
- a collection $\mathcal{G} = \{G_e\}_{e \in \mathcal{E}}$ of *guards* $G_e \subset D_{\mathfrak{s}e} \times \mathbb{T}$,
- and a collection $\mathcal{R} = \{R_e\}_{e \in \mathcal{E}}$ of *resets* $R_e \subset G_e \times D_{\mathfrak{t}e} \times \mathbb{T}$.

The *hybrid phase space* $D \subset \Lambda \times \mathbb{R}^n$ of a hybrid dynamical system $\mathscr{H}$ is given by

$$D = \bigcup_{\lambda \in \Lambda} \mathcal{D}(\lambda) \subset \Lambda \times \mathbb{R}^n, \tag{2.2}$$

which projects surjectively onto Λ by definition. For the sake of simplicity, we

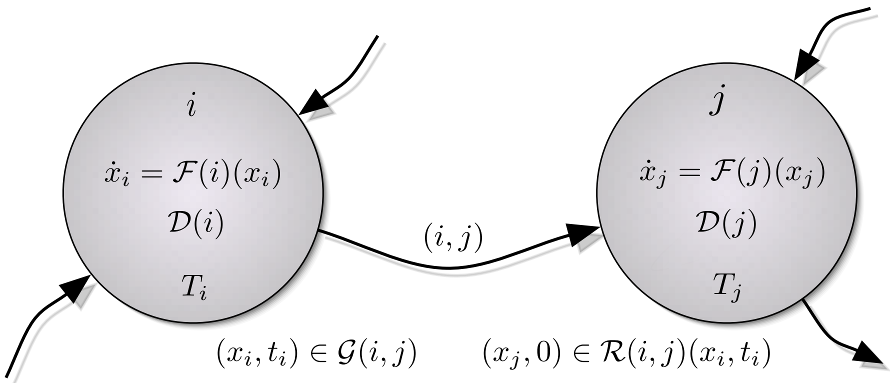


Figure 2.1: Graphical representation of a hybrid automaton $\mathscr{H}$ based on [SJLS05]: The spheres correspond to individual dynamical systems, while the arrows indicate admissible transitions from one system to another.

will not distinguish between the set $D_\lambda \subset \{\lambda\} \times \mathbb{R}^n$ and its projection onto $\mathbb{R}^n$.

The formal structure as well as the functionality of a general hybrid dynamical system $\mathscr{H}$ can be briefly highlighted in the following manner: The foundation is made up by a *discrete transition graph* $\mathscr{T} = (\Lambda, \mathcal{E})$ where Λ denotes the set of discrete states of the system and $\mathcal{E}$ the admissible transitions between these states. The hybrid system's continuous dynamics ultimately result from the assignment of a dynamical system $\dot{x} = f_\lambda(x)$ determined by $\Psi_\lambda = (D_\lambda, f_\lambda)$ with $D_\lambda \subset \{\lambda\} \times \mathbb{R}^n$ to each discrete state $\lambda \in \Lambda$ in virtue of the *phase space and vector field families* $\mathcal{D}$ and $\mathcal{F}$, respectively. This discrete state-wise introduction of dynamics results in a family of dynamical systems abstractly connected by the discrete transitions $\mathcal{E}$, which have to be enriched by conditions that decide when discrete transitions are enabled and in which manner they are realized. These tasks are performed by the *guards* and the *resets*: The guards $\mathcal{G}(e) = G_e$ activate possible transitions between discrete states with regard to their internal dynamical system structure, whereas the resets R_e provide the necessary initial conditions after switching discrete states in order to maintain the further evolution of the system as a whole. The clocks collected in $\mathcal{C}$ provide the possibility of triggering discrete state transitions that are exclusively directed by time.

Commonly, hybrid dynamical systems are graphically represented in the following style: The discrete states $\lambda \in \Lambda$ are depicted in form of labeled spheres connected by arrows which symbolize the discrete transitions $e \in \mathcal{E}$. Each sphere corresponding to a discrete state λ is equipped with the information about the dynamical system corresponding to this state, i. e. the phase space $\mathcal{D}(\lambda)$ and the vector field $\mathcal{F}(\lambda)$ and optionally – for the sake of completeness – the accordant clock T_λ. Finally, nearby its source and tail every arrow representing a transition $e \in \mathcal{E}$ is decorated by the guard data $\mathcal{G}(e)$ and the reset data $\mathcal{R}(e)$, respectively. See Figure 2.1 for a prototypical visualization of a hybrid dynamical system.

2.1.2 Remark (On the Nature of Guards and Resets). For convenience in handling guards and resets in presence of clocks which definitely do not play any role for the internal dynamics of the various discrete states, we denote by $D_{\mathbb{T},\lambda}$ the extended phase space $D_\lambda \times \mathbb{T}$. Note that in Definition 2.1.1 the resets $\mathcal{R}(e) = R_e \subset G_e \times D_{\mathbb{T},\mathfrak{t}e}$, $e \in \mathcal{E}$, are formally given as graphs of generally set-valued maps $R_e : G_e \to D_{\mathbb{T},\mathfrak{t}e}$, which may serve as a source of *non-determinism*. For reasons of consistency, the resets $\mathcal{R}(e) \subset \mathcal{G}(e) \times D_{\mathbb{T},\mathfrak{t}e}$ project surjectively onto the guards, i. e.

$$(\mathcal{R}(e))_{\mathbb{T},\mathfrak{s}e} = \mathcal{G}(e) \quad \text{for all } e \in \mathcal{E}, \tag{2.3}$$

with $X_{\mathbb{T},\lambda}$, $\lambda \in M \subset \Lambda$, denoting the image of a set $X \subset \prod_{\mu \in M \subset \Lambda} D_{\mathbb{T},\mu}$ under the projection $\mathrm{pr}_\lambda : \prod_{\mu \in M} D_{\mathbb{T},\mu} \twoheadrightarrow D_{\mathbb{T},\lambda}$. Consequently, from the first, the guard information is included in the resets and can always be extracted from $\mathcal{R}$. By means of this observation, it is not essential to explicitely deal with the guards in case one has access to the reset information, but there are imaginable situations where discussing guards and resets separately may be highly convenient. In this context, we are forced to assume the non-emptiness of guards and of resets, namely $\mathcal{G}(e) \neq \emptyset$ and of resets, namely $\mathcal{R}(e)(x) \neq \emptyset$ for all $e \in \mathcal{E}$ and $(x_{\mathfrak{s}e}, t_{\mathfrak{s}e}) \in \mathcal{G}(e)$. What is more, with a view to clocks, the resets are to reset the time component to zero, i.e. $R_e(x_{\mathfrak{s}e}, t_{\mathfrak{s}e}) = (x_{\mathfrak{t}e}, 0_{\mathfrak{t}e})$, in order to ensure that the clock of the current discrete state precisely shows how long the system has remained in that state after the preceding transition. $\diamondsuit$

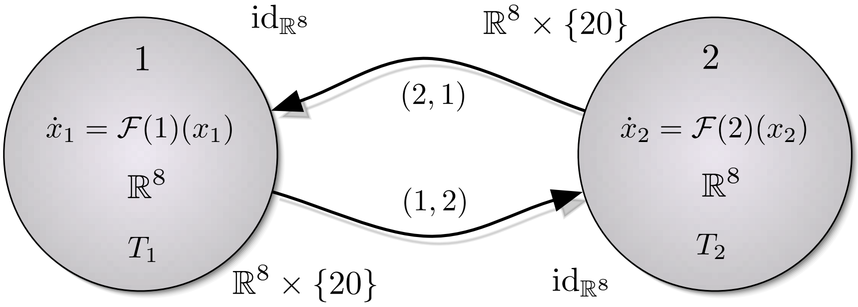


Figure 2.2: Example of a simple hybrid dynamical system with time-dependent switching and trivial resets; hybrid dynamical systems of that kind are to be discussed in Section 5.1 in more detail.

2.1.3 Example. Let the hybrid dynamical system $\mathscr{H} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{C}, \mathcal{G}, \mathcal{R})$ be given by the simple transition graph $\mathscr{T} = (\Lambda, \mathcal{E})$ with

$$\Lambda = \{1, 2\} \quad \text{and} \quad \mathcal{E} = \{(1,2), (2,1)\}$$

and the eight-dimensional dynamical systems determined by the vector fields $\mathcal{F}(\lambda) : \mathbb{R}^8 \to \mathbb{R}^8$, $\lambda = 1, 2$, which are given by

$$\mathcal{F}(1)(x) = \begin{pmatrix} -(0.39+\zeta)x_1 - 0.4x_2 + \zeta x_3 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39+\zeta)x_2 + \zeta x_4 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39+2\zeta)x_3 - 0.4x_4 + \zeta x_5 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39+2\zeta)x_4 + \zeta x_6 + 2.5\epsilon x_3 x_4 \\ \zeta x_3 - (0.39+2\zeta)x_5 - 0.4x_6 + \zeta x_7 + \epsilon x_5 x_6^2 \\ \zeta x_4 + 0.04x_5 - (0.39+2\zeta)x_6 + \zeta x_8 + 2.5\epsilon x_5 x_6 \\ \zeta x_5 - (0.39+\zeta)x_7 - 0.4x_8 + \epsilon x_7 x_8^2 \\ \zeta x_6 + 0.04x_7 - (0.39+\zeta)x_8 + 2.5\epsilon x_7 x_8 \end{pmatrix} \tag{2.4}$$

and

$$\mathcal{F}(2)(x) = \begin{pmatrix} -(0.39+2\zeta)x_1 - 0.4x_2 + \zeta x_3 + \zeta x_7 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39+2\zeta)x_2 + \zeta x_4 + \zeta x_8 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39+\zeta)x_3 - 0.4x_4 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39+\zeta)x_4 + 2.5\epsilon x_3 x_4 \\ -(0.39+\zeta)x_5 - 0.4x_6 + \zeta x_7 + \epsilon x_5 x_6^2 \\ 0.04x_5 - (0.39+\zeta)x_6 + \zeta x_8 + 2.5\epsilon x_5 x_6 \\ \zeta x_1 + \zeta x_5 - (0.39+2\zeta)x_7 - 0.4x_8 + \epsilon x_7 x_8^2 \\ \zeta x_2 + \zeta x_6 + 0.04x_7 - (0.39+2\zeta)x_8 + 2.5\epsilon x_7 x_8 \end{pmatrix} \tag{2.5}$$

with $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T \in \mathbb{R}^8$ and real parameters $\zeta, \epsilon \in \mathbb{R}$. The guards are given by $\mathcal{G}(1,2) = \mathcal{G}(2,1) = \mathbb{R}^8 \times \{20\}$ and the resets are trivial, i. e. $\mathcal{R}(1,2) = \mathcal{R}(2,1) \cong \Delta(\mathbb{R}^8 \times \mathbb{R}^8) \times \{20\} \times \{0\}$, or simply, $\mathcal{R}(e) = \mathrm{id}_{\mathbb{R}^8} \times 0$ when understanding them as maps on $\mathcal{G}(e)$. Figure 2.2 provides a graphical visualization of the hybrid dynamical system $\mathscr{H}$. $\diamondsuit$

2.2 Hybrid Dynamics

On the grounds of Definition 2.1.1, we address the dynamic nature of hybrid dynamical systems. The general mechanism of hybrid dynamical behavior works as follows: The current state $p = (\lambda, x) \in D$ of a hybrid dynamical system consists of its discrete state $\lambda \in \Lambda$ and its continuous state $x \in \mathbb{R}^n$. Being in discrete state λ the system evolves according to the active dynamical system $\dot{x}_\lambda = \mathcal{F}(\lambda)(x_\lambda)$ until the trajectory x_λ happens to hit a guard $\mathcal{G}(e)$ with $\mathfrak{s}e = \lambda$ which is a subset of the phase space $\mathcal{D}(\lambda)$. As soon as a guard is met, a discrete transition to the discrete state $\mathfrak{t}e$ is enabled; in case the hybrid dynamical system is *deterministic*, this transition *has* to take place precisely at the moment of meeting the guard condition. The task of the resets is then to reset the discrete state to $\mathfrak{t}e$, provide a new continuous initial condition in $\mathbb{R}^n$ and to set back the time to zero.

In the following, we give a precise definition of hybrid trajectories serving as an analogy to the classical notion of solution with respect to a dynamical system $\dot{x} = f(x)$. In consequence of the interlocked discrete and continuous characteristics of hybrid dynamical systems, the formulation of an according solution or trajectory concept requires an adapted notion of time which reflects the composite structure of the system. The central issue of this adaptation consists in the administration of discrete transitions.

2.2.1 Definition (Hybrid Time Trajectory, cf. [SJLS05]). A *hybrid time trajectory* (or, more plainly, *hybrid time set*) τ is a finite or infinite sequence $\{I_k\}_{k=0}^{N}$, $N \in \mathbb{N}_0 \cup \{\infty\}$, of intervals I_k with

- $I_k = [\tau_k, \tau'_k]$ for all $k < N$,
- $\tau_k \leq \tau'_k = \tau_{k+1}$ for all k,
- in case $N < \infty$ either $I_N = [\tau_N, \tau'_N]$ or $I_N = [\tau_N, \tau'_N)$. ♢

By $|I_k| = \tau'_k - \tau_k$ we denote the length of the interval I_k and by $\mathfrak{T}$ the set of hybrid time trajectories. The *discrete length* $\langle\cdot\rangle$ of a hybrid time trajectory $\tau = \{I_k\}_{k=0}^{N}$ is defined by the map

$$\langle\cdot\rangle : \mathfrak{T} \longrightarrow \mathbb{N}_0 \cup \{\infty\}\,, \qquad \langle\tau\rangle := N, \tag{2.6}$$

and its *continuous length* is determined by

$$(\cdot) : \mathfrak{T} \longrightarrow \mathbb{R}_+ \cup \{\infty\}\,, \qquad (\tau) := \sum_{k=0}^{\langle\tau\rangle} |I_k|. \tag{2.7}$$

The discrete length $\langle\cdot\rangle$ gives rise to a set-valued map

$$[\cdot] : \mathfrak{T} \to 2^{\mathbb{N}_0}, \qquad [\tau] := \begin{cases} [\langle\tau\rangle] = \{0, \dots, \langle\tau\rangle\} & \text{if } \langle\tau\rangle < \infty \\ \mathbb{N}_0 & \text{else.} \end{cases}$$

The time instants τ_k of a hybrid time trajectory correspond to switching times at which discrete transitions take place. Thus, the discrete length $\langle\tau\rangle$ offers the number of discrete transitions with respect to the hybrid time trajectory τ, while the continuous length (τ) measures the time of flowing.

2.2.2 Definition (Execution, cp. [ZJLS00]). An *execution* of a hybrid dynamical system $\mathscr{H}$ is a triple $\chi = (\tau, \gamma, x)$ where

- $\tau \in \mathfrak{T}$ is a hybrid time trajectory,
- $\gamma : [\tau] \to \Lambda$ is a map encoding the visited discrete states,

- $x = \{x_k\}_{k\in[\tau]}$ is a collection of C^1-maps such that $x_k : I_k \to \mathcal{D}(\gamma(k))$ is a trajectory of the corresponding dynamical system $\dot{x}_{\gamma(k)} = \mathcal{F}(\gamma(k))(x_{\gamma(k)})$

such that for all $k \in [\tau] \setminus \langle\tau\rangle$ the following conditions hold:

- Admissibility of the discrete transition from $\gamma(k)$ to $\gamma(k+1)$:

$$(\gamma(k), \gamma(k+1)) \in \mathcal{E}, \tag{2.9}$$

- Guard compatibility:

$$(x_k(\tau'_k), |I_k|) \in G_{(\gamma(k),\gamma(k+1))} \subset D_{\mathbb{T},\gamma(k)}, \tag{2.10}$$

- Reset compatibility:

$$\big((x_k(\tau'_k), |I_k|), (x_{k+1}(\tau_{k+1}), 0_{\gamma(k+1)})\big) \in R_{(\gamma(k),\gamma(k+1))}. \tag{2.11}$$

For $(\lambda_0, x_0) \in D$, we denote by $E_{(\lambda_0,x_0)}$ the set of all executions of $\mathscr{H}$ starting in (λ_0, x_0). Furthermore, we set

$$E = \bigcup_{(\lambda_0,x_0)\in D} E_{(\lambda_0,x_0)} \tag{2.12}$$

to obtain a set including *all* executions of $\mathscr{H}$. In order to account for the possible non-determinism of a hybrid dynamical system, the manner of speaking is that a hybrid system $\mathscr{H}$ *accepts* an execution χ if it is in line with Definition 2.2.2. Essentially, an execution is a pair of maps (γ, x) governing the discrete and continuous dynamical behavior, respectively: Here, on the discrete level, γ outputs feasible discrete transitions at the transition times τ_k and, simultaneously, on the continuous level, the dynamical systems' trajectories gathered by x are in accordance with the guard and reset data contained in the system. An exemplary plot of an execution generated by a hybrid dynamical system as shown in Figure 2.2; the bold line segments and shading record the switching behavior of the system. Analogous to trajectories of classical dynamical systems, we can translate an execution χ given as a triple (τ, γ, x) to the more familiar form $\chi : \mathcal{T} \to D$ with D denoting the global hybrid phase space as given in (2.2) and $\mathcal{T}$ a hybrid version of time. For this purpose, we define the

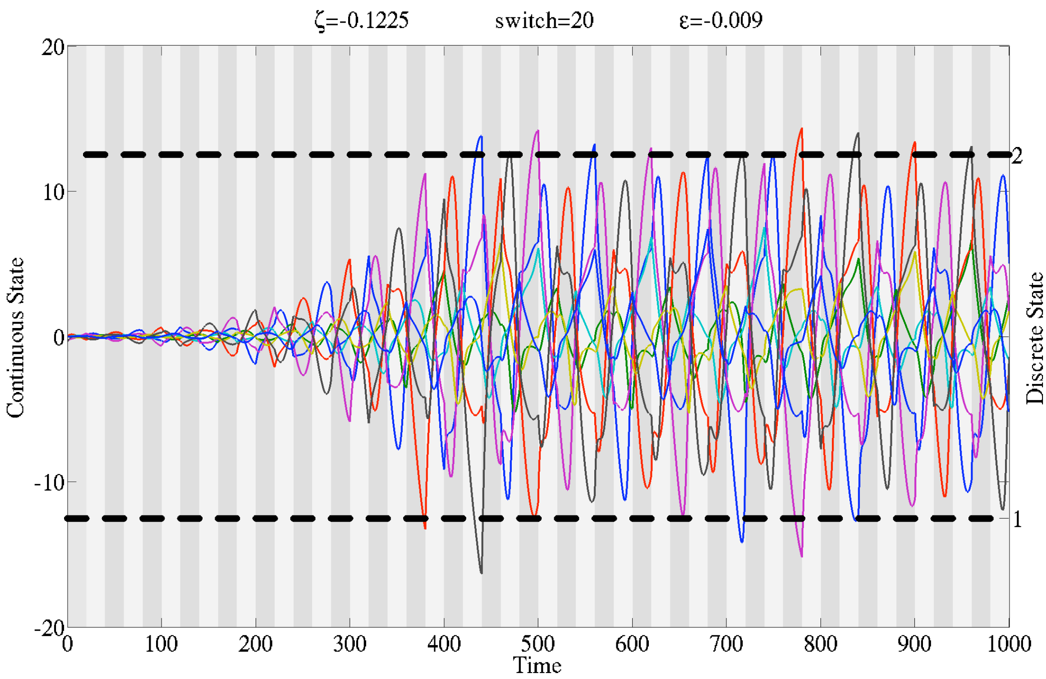


Figure 2.3: Execution of an 8-dimensional hybrid dynamical system (of the kind as illustrated in Fig. 2.2) with each coordinate plotted separately; the bold line segments as well as the shading visualize the underlying hybrid time trajectory.

hybrid time $\mathcal{T} = \mathcal{T}_\tau$ induced by the hybrid time set $\tau = \{I_k\}_{k\in\mathbb{N}}$ as follows: For a closed interval $I = [a,b]$, we set

$$I^{)} = \begin{cases} [a,b) & \text{if } a \neq b \\ \{a\} & \text{if } a = b \end{cases}$$

and $I^{)} = I$ in case I is right-open. Herewith, we define

$$\mathcal{T} = \bigcup_{k\in[\tau]} \{k\} \times I_k^{)} \subset \mathbb{Z} \times \mathbb{R}. \tag{2.13}$$

Now, an execution $\chi = (\tau, \gamma, x)$ can be written in the form

$$\chi : \mathcal{T}_\tau \to D, \quad (k,t) \mapsto \chi(k,t) = (\gamma(k), x_k(t)) \,. \tag{2.14}$$

Note that $\mathcal{T}$ does not exhibit obvious algebraic structure meaning that $\mathcal{T}$ is in general no subgroup of $\mathbb{Z} \times \mathbb{R}$ and does not even possess the structure of a semigroup. The *image* of an execution χ is given by

$$\mathcal{I}_\chi = \{\chi(k,t)\}_{(k,t)\in\mathcal{T}_\tau} \subset D. \tag{2.15}$$

By means of the discrete and continuous length of hybrid time trajectories (see (2.6) and (2.7)), the executions of a hybrid dynamical system $\mathscr{H}$ can be classified in the following manner (cf. [LJSE99], for instance):

2.2.3 Definition (Classification of Executions, see [LJSE99]). An execution $\chi = (\tau, \gamma, x)$ of a hybrid dynamical system $\mathscr{H}$ is called

- *finite*, if $\langle\, \tau \,\rangle < \infty$ and the interval $I_{\langle\, \tau \,\rangle}$ is closed,
- *infinite*, if $\langle\, \tau \,\rangle = \infty$ or $(\tau) = \infty$,
- *Zeno*, if $\langle\, \tau \,\rangle = \infty$, but $(\tau) < \infty$. ◇

An exotic and occasionally unwanted type of hybrid dynamics is generated by Zeno executions which perform infinitely many discrete transitions in finite time. Zeno behavior is highly unphysical but it may arise at times due to modelling abstraction. The class of *Zeno hybrid systems* which admit such undesirable solutions is studied in [ZJLS00] and [ZJLS01]; furthermore, [JELS99] proposes and analyzes regularization techniques which can be applied in order to overcome Zeno dynamics.

Necessary and sufficient conditions for the existence and uniqueness of executions are treated in [LJSE99] and [LJS$^+$03]; moreover, [LJS$^+$03] additionally discusses the continuous dependence on initial conditions. In combination with the above presented classification of executions (together with a notion of maximality), the results of these articles distinguish two important classes of hybrid dynamical systems: the *deterministic* and the *non-blocking* ones.

2.2.4 Definition (Non-Blockingness and Determinism, cf. [LJSE99]). A hybrid dynamical system $\mathscr{H}$ is *non-blocking* if for every $p_0 = (\lambda_0, x_0) \in D$ there exists at least one infinite execution χ starting at p_0. $\mathscr{H}$ is called *deterministic* if for every $p_0 \in D$ there is at most one maximal execution χ starting at p_0, where maximality means that there is no execution $\widehat{\chi}$ such that the hybrid time set τ is a prefix of $\widehat{\tau}$, i. e. $\langle\tau\rangle \leq \langle\widehat{\tau}\rangle$, $I_k = \widehat{I}_k$ for all $k \in [\tau]$ and $I_{\langle\tau\rangle} \subseteq \widehat{I}_{\langle\tau\rangle}$ and that $\chi(k,t) = \widehat{\chi}(k,t)$ for all $(k,t) \in \mathcal{T}$. $\diamondsuit$

In practical applications, when analysis and modeling of hybrid dynamical systems is concerned, mainly the *finite* executions play a crucial role, since they encode the information which hybrid states are *reachable* by a hybrid system in *finite* time. This set of states termed the *reachability domain* is of major importance, for instance, whenever *safety properties* are inspected.

2.2.5 Definition (Reachability Domain, cp. [LJSE99]). Given a hybrid dynamical system $\mathscr{H}$ with hybrid phase space D, let $E^{<\infty} \subset E$ denote the set of all finite executions of $\mathscr{H}$. For each $\chi \in E^{<\infty}$ there exists a uniquely determined final state, denoted by $\odot(\chi)$. The collection $\mathrm{Reach}(\mathscr{H}) \subset D$ of all finite-time reachable states is given by

$$\begin{aligned} \mathrm{Reach}(\mathscr{H}) &= \{(\lambda, x) \in D \mid \exists\, \chi \in E^{<\infty} : \odot(\chi) = (\lambda, x)\} \qquad (2.16) \\ &= \{\odot(\chi) \in D \mid \chi \in E^{<\infty}\}. \end{aligned}$$

and referred to as the *reachability domain* of $\mathscr{H}$. $\diamondsuit$

Again in connection with safety properties a specific class of sets is of immense importance, namely the sets that are invariant under the dynamics of a hybrid dynamical system.

2.2.6 Definition (Hybrid Invariant Set, cp. [LJSE99] or [ZJLS00]). Let $\mathscr{H}$ be a hybrid dynamical system. A subset $\mathcal{S} \subset D$ is *invariant* (in the hybrid sense) if every execution χ of $\mathscr{H}$ starting in $\mathcal{S}$ always stays inside $\mathcal{S}$. $\diamondsuit$

For clarity, it is important to stress here that invariance in the classical sense of all dynamical systems involved is in general not sufficient to guarantee hybrid invariance: Even when $\mathcal{S}_\lambda$ is classically invariant for *every* discrete state $\lambda \in \Lambda$, an execution may exit $\mathcal{S}$ via a guard $\mathcal{G}(e)$ with $\mathfrak{t}e \notin \widehat{\mathcal{S}}$. This circumstance gains special importance as soon as hybrid fixed-point spaces are treated and their invariance is analyzed in Section 3.5.

Chapter 3

Hybrid Symmetries

Dynamical systems in the usual sense, i.e. ordinary differential equations in the case of continuous time or difference equations in the case of discrete time, have gathered a broad and well-developed theory of symmetry around themselves: [Fie88], [GSS88], [HLB96] and [GS02] to name just a few but extremely prominent works on symmetry.

However, looking through the literature's glass, one cannot verify an analogous statement for hybrid systems. That is to say that up to now there is seemingly no fundamental systematic approach to the formal comprehension and treatment of symmetry properties arising in the regime of hybrid systems. Nevertheless, some points of contact can be located: In [HS02], where the optimal control problem of switched Lagrangian systems is studied in presence of a single group of symmetries acting on all phase spaces involved while all system components are invariant under this action. The symmetries considered there thus do not have any hybrid traits, in particular the discrete part of the composite system structure does not enter the treatment of symmetries at all. In [BK08], symmetry reduction for stochastic hybrid systems is considered; again, the considered type of symmetry is not hybrid in the sense as discussed in this thesis. In [RMMC02], periodically forced chemical reactors are examined and symmetry properties related to the switching are detected. Modelled as a cyclic network of *identical* systems subjected to discontinuous forcing, the global system is hybrid, but the occuring spatio-temporal patterns are not at all analyzed with regard to the overall structure of the hybrid system. Since the components involved are all identical, symmetries simply arise from the periodic nature of the forcing. Again, the symmetries occuring are not strictly hybrid but exclusively induced by time.

In the face of these circumstances, the aim of this chapter is to overcome this lack of elementary theory, to develop an abstract concept of hybrid symmetries and thereby to provide an adequate framework to handle such symmetries and to analyze their effect on the dynamics of hybrid systems.

The roadmap for this project is as follows: To begin with, Section 3.1 recalls the classical concept of symmetries for dynamical systems $\dot{x} = F(x)$ and briefly sketches structural aspects as well as immediate consequences on the dynamics. After that, Section 3.2 lays the groundwork for hybrid symmetries by considering the abstract predecessors of hybrid dynamical systems termed *dynamical $\mathscr{T}$-systems* which could carefully be viewed as prototypes for any kind of coupled systems. A global notion of symmetry represented by the *$\mathscr{T}$-symmetries* is constructed and examined from a structural point of view. Based thereupon, Section 3.3 pictures the way from dynamical $\mathscr{T}$-systems to hybrid dynamical systems and extends $\mathscr{T}$-symmetries to hybrid symmetries creating a concept of global symmetries for hybrid dynamical systems. It is Section 3.4 that sheds light on the immediate consequences of hybrid symmetries on the dynamics; the findings of this section in turn argue for the conceptual adequacy of hybrid symmetries. Finally, in Section 3.5 symmetry-induced *hybrid fixed-point spaces* are discussed with a particular emphasis on their invariance properties which prove to be considerably weaker than in the classical setting of equivariant dynamical systems. This last mentioned fact actually constitutes a major difference between hybrid symmetries and classical symmetries.

3.1 Classical Symmetries of Dynamical Systems

This section serves as a guide to the formalism of symmetries in dynamical system theory. We mainly refer to [GSS88] and [GS02]. Let $\Psi = (X, F)$ be a dynamical system with phase space $X \subset \mathbb{R}^n$ and vector field $F : X \to \mathbb{R}^n$ (at least continuously differentiable). Here, Ψ is meant to be understood as the ordinary differential equation $\dot{x} = F(x)$.

3.1.1 Definition. A *symmetry* of a dynamical system $\Psi = (X, F)$ is an element $\gamma \in O(n)$ which commutes with F, i.e. $\gamma \circ F = F \circ \gamma$. In this case, F is also said to be *equivariant* with respect to γ. $\diamondsuit$

For a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ the unit matrix $\mathbb{I}_n \in O(n)$ is always a symmetry since it trivially fulfills the equivariance condition. Before we proceed with a characterization of symmetries in terms of dynamics, we examine a characteristic example which is of particular importance for the course of this thesis and which we will therefore encounter again in later chapters.

3.1.2 Example. For $\zeta \in \mathbb{R}$, we consider the eight-dimensional dynamical system $\Psi = (\mathbb{R}^8, F_\zeta)$ with vector field $F_\zeta : \mathbb{R}^8 \to \mathbb{R}^8$ given by

$$F_\zeta(x) = \begin{pmatrix} -(0.39+2\zeta)x_1 - 0.4x_2 + \zeta x_3 + \zeta x_7 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39+2\zeta)x_2 + \zeta x_4 + \zeta x_8 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39+2\zeta)x_3 - 0.4x_4 + \zeta x_5 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39+2\zeta)x_4 + \zeta x_6 + 2.5\epsilon x_3 x_4 \\ \zeta x_3 - (0.39+2\zeta)x_5 - 0.4x_6 + \zeta x_7 + \epsilon x_5 x_6^2 \\ \zeta x_4 + 0.04x_5 - (0.39+2\zeta)x_6 + \zeta x_8 + 2.5\epsilon x_5 x_6 \\ \zeta x_1 + \zeta x_5 - (0.39+2\zeta)x_7 - 0.4x_8 + \epsilon x_7 x_8^2 \\ \zeta x_2 + \zeta x_6 + 0.04x_7 - (0.39+2\zeta)x_8 + 2.5\epsilon x_7 x_8 \end{pmatrix}$$

and $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T \in \mathbb{R}^8$. Let us consider the permutation group S_8 acting on $\mathbb{R}^8$ via

$$gx = \left(x_{g^{-1}(1)}, x_{g^{-1}(2)}, x_{g^{-1}(3)}, x_{g^{-1}(4)}, x_{g^{-1}(5)}, x_{g^{-1}(6)}, x_{g^{-1}(7)}, x_{g^{-1}(8)}\right).$$

Let $g = (1357)(2468) \in S_8$. Then the action of g on the state x can be described in terms of the matrix

$$\rho(g) = \begin{pmatrix} 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix} \in O(8).$$

We now compute

$$\rho(g)F_\zeta(x) = \begin{pmatrix} \zeta x_1 + \zeta x_5 - (0.39+2\zeta)x_7 - 0.4x_8 + \epsilon x_7 x_8^2 \\ \zeta x_2 + \zeta x_6 + 0.04x_7 - (0.39+2\zeta)x_8 + 2.5\epsilon x_7 x_8 \\ -(0.39+2\zeta)x_1 - 0.4x_2 + \zeta x_3 + \zeta x_7 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39+2\zeta)x_2 + \zeta x_4 + \zeta x_8 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39+2\zeta)x_3 - 0.4x_4 + \zeta x_5 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39+2\zeta)x_4 + \zeta x_6 + 2.5\epsilon x_3 x_4 \\ \zeta x_3 - (0.39+2\zeta)x_5 - 0.4x_6 + \zeta x_7 + \epsilon x_5 x_6^2 \\ \zeta x_4 + 0.04x_5 - (0.39+2\zeta)x_6 + \zeta x_8 + 2.5\epsilon x_5 x_6 \end{pmatrix}$$

and

$$\begin{aligned} F_\zeta(\rho(g)x) &= F_\zeta(x_7, x_8, x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \begin{pmatrix} -(0.39+2\zeta)x_7 - 0.4x_8 + \zeta x_1 + \zeta x_5 + \epsilon x_7 x_8^2 \\ 0.04x_7 - (0.39+2\zeta)x_8 + \zeta x_2 + \zeta x_6 + 2.5\epsilon x_7 x_8 \\ \zeta x_7 - (0.39+2\zeta)x_1 - 0.4x_2 + \zeta x_3 + \epsilon x_1 x_2^2 \\ \zeta x_8 + 0.04x_1 - (0.39+2\zeta)x_2 + \zeta x_4 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39+2\zeta)x_3 - 0.4x_4 + \zeta x_5 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39+2\zeta)x_4 + \zeta x_6 + 2.5\epsilon x_3 x_4 \\ \zeta x_7 + \zeta x_3 - (0.39+2\zeta)x_5 - 0.4x_6 + \epsilon x_5 x_6^2 \\ \zeta x_8 + \zeta x_4 + 0.04x_5 - (0.39+2\zeta)x_6 + 2.5\epsilon x_5 x_6 \end{pmatrix}. \end{aligned}$$

Thus we find the equivariance $\rho(g)F_\zeta(x) = F_\zeta(\rho(g)x)$ for all $x \in \mathbb{R}^8$. Hence, $\rho(g)$ turns out to be a symmetry of the dynamical system $\dot{x} = F_\zeta(x)$ for every choice of ζ. $\diamondsuit$

The definition of symmetries as above triggers two important results: One result concerns the dynamical behavior of the inspected dynamical system, and the other one uncovers the algebraic structure of the collection of all symmetries of Ψ.

3.1.3 Theorem (Dynamical Characterization of Symmetries, [GS02]). *An element $\gamma \in O(n)$ is a symmetry of the dynamical system $\Psi = (X, F)$ in the sense of Definition 3.1.1 if and only if it transforms solutions into solutions: For every solution $x(t)$ of Ψ, $\gamma x(t)$ is a solution of Ψ, as well.*

This is due to the reason that one has $\dot{gx}(t) = g\dot{x}(t) = gF(x(t)) = F(gx(t))$ for a solution $x(t)$ of $\Psi = (X, F)$. Essentially, this theorem states that symmetries of dynamical systems are compatible with the dynamics of the system while leaving the set of trajectories invariant. An important example building on Example 3.1.2 is provided below.

3.1.4 Example. Let $x_0 \in \mathbb{R}^8$ be an equilibrium of $\Psi_\zeta = (\mathbb{R}^8, F_\zeta)$ for some $\zeta \in \mathbb{R}$, i. e. $F_\zeta(x_0) = 0$. Then, we know by Theorem 3.1.3 that gx_0 is also a solution of Ψ_ζ for a symmetry g of Ψ_ζ. According to the equivariance condition gx_0 is in fact another equilibrium of Ψ_ζ. For instance, we observe that the origin $x_0 = 0 \in \mathbb{R}^8$ is an equilibrium of $\Psi_\zeta = (\mathbb{R}^8, F_\zeta)$ for every choice of ζ since $F_\zeta(x_0) = 0$. In particular, we have $gx_0 = x_0$ for all $g \in S_8$ in this case. $\diamondsuit$

The collection of all symmetries of a given dynamical system Ψ is well-known to possess algebraic structure making equivariant dynamical systems amenable to algebraic analysis.

3.1.5 Theorem (Algebraic Structure of Symmetries, [GSS88]). *The collection G of all symmetries of the dynamical system $\Psi = (X, F)$ forms a group. It is termed the* symmetry group *of Ψ.*

The pair (Ψ, G) is said to be an *equivariant dynamical system* and together with Theorem 3.1.3 and Definition 3.1.1, we see that the vector field F is G-equivariant. This structural statement especially ensures that whenever a specific (non-trivial) symmetry of order ≥ 2 of a dynamical system is known, we are assured that a whole (non-trivial) group of symmetries does exist.

3.1.6 Example. In Example 3.1.2, we have discovered the symmetry $g = (1357)(2468) \in S_8$ with representing matrix $\rho(g) \in \mathbb{R}^{8\times 8}$ of the dynamical system $\Psi_\zeta = (\mathbb{R}^8, F_\zeta)$. In a situation like this, Theorem 3.1.5 tells us that all iterates g^k, $k \in \mathbb{Z}$, of g are symmetries of Ψ_ζ or – differently speaking – the cyclic group $\langle g \rangle \leq S_8$ generated by g is a subgroup of the symmetry group G of Ψ_ζ. Observe that $\mathrm{ord}(g) = 4$. Thus, $\langle g \rangle \cong \mathbb{Z}_4$ and hence Ψ_ζ is seen to exhibit at least the non-trivial symmetries $g = (1357)(2468)$, $g^2 = (15)(26)(37)(48)$ and $g^3 = (1753)(2864)$ with $\rho(g^2) = \rho(g)^2$ and $\rho(g^3) = \rho(g)^3$. $\diamondsuit$

An immediate consequence for the system's dynamics brought about by symmetries is the generation of special *flow-invariant subspaces*, which are regions of the phase space that – once entered by a trajectory – can never be left again.

3.1.7 Definition (Fixed-Point Spaces, see [GS02]). Let (Ψ, G) be an equivariant dynamical system. For a subgroup $\Sigma \leq G$ define

$$\mathrm{Fix}(\Sigma) := \{x \in X \mid \sigma x = x \text{ for all } \sigma \in \Sigma\}. \tag{3.1}$$

This subset of X is referred to as the *fixed-point space* of Σ. $\diamondsuit$

Note that fixed-point spaces are indeed subspaces since the equation

$$\mathrm{Fix}(\Sigma) = \bigcap_{\sigma \in \Sigma} \ker(\mathrm{id} - \sigma) \tag{3.2}$$

holds. Furthermore, the relation $\mathrm{Fix}(g\Sigma g^{-1}) = g\mathrm{Fix}(\Sigma)$ is valid for a subgroup $\Sigma \leq G$ and an element $g \in G$ (cp. [GS02]).

3.1.8 Example. Let us reconsider the dynamical system $\Psi_\zeta = (\mathbb{R}^8, F_\zeta)$ as in the preceding examples. By Σ_g we denote the cyclic group $\langle g \rangle \leq S_8$. We aim to examine the fixed-point subspace $\mathrm{Fix}(\Sigma_g)$. For this purpose, we note that the group of permutations Σ_g partitions $\{1, \ldots, 8\}$ into odd and even numbers since $gx = x$ implies $x_1 = x_7 = x_3 = x_5$ and $x_2 = x_4 = x_6 = x_8$. Consequently, we obtain $\mathrm{Fix}(\Sigma_g) = \{(x, y, x, y, x, y, x, y) \mid x, y \in \mathbb{R}\}$ revealing $\dim(\mathrm{Fix}(\Sigma_g)) = 2$. Notably, this subspace describes the set of states $x \in \mathbb{R}^8$ whose odd and even coordinates are precisely behaving the same way, respectively, that is to say that they are *in synchrony*. This is why fixed-point spaces are also often called *synchrony subspaces*. $\diamondsuit$

As already indicated above, fixed-point subspaces are characterized by their flow-invariance.

3.1.9 Theorem (Flow-Invariance of Fixed-Point Spaces, [GS02]). *Fixed-point spaces are flow-invariant: For an equivariant dynamical system (Ψ, G) with $\Psi = (X, F)$ and any subgroup $\Sigma \leq G$, one has*

$$F(\mathrm{Fix}(\Sigma)) \subset \mathrm{Fix}(\Sigma). \tag{3.3}$$

The proof of this meaningful statement is surprisingly simple: For $x \in \mathrm{Fix}(\Sigma)$, one has $F(x) = F(gx) = gF(x)$ for all $g \in \Sigma$ and hence $F(x) \in \mathrm{Fix}(\Sigma)$.

Observe that this statement implies that a G-equivariant vector field F restricts to maps

$$F|_{\mathrm{Fix}(\Sigma)} : \mathrm{Fix}(\Sigma) \to \mathrm{Fix}(\Sigma)$$

for every subgroup $\Sigma \leq G$. This is a significant statement since the fixed-point spaces $\mathrm{Fix}(\Sigma)$ may be of lower dimension than the whole space X such that the task of finding equilibria with prescribed symmetries, for instance, becomes easier. In this context, the symmetries of a state x are described by its *isotropy subgroup*.

3.1.10 Definition (Isotropy Subgroup, [GS02]). Let $x \in X$ be a state of an equivariant dynamical system (Ψ, G). The *isotropy subgroup* of x is determined by

$$\Sigma_x := \{g \in G \mid gx = x\} \leq G. \qquad \diamondsuit$$

Observe that states on the same orbit G_{x_0} have conjugate isotropy group, i. e. for $g \in G$, we have $\Sigma_{gx} = g\Sigma_x g^{-1}$ for every state x (cf. [GSS88]). Moreover, Theorem 3.1.9 implies the constancy of symmetries along trajectories.

3.1.11 Corollary (Isotropy Subgroups of Trajectories, [GS02]). *Isotropy subgroups remain constant along trajectories: If $x : I \to X$ is a solution of (Ψ, G), then $\Sigma_{x(0)} = \Sigma_{x(t)}$ for all $t \in I$.*

The preceding collection of definitions and results provide a foundation for the treatment of symmetries in the field of dynamical systems. We will make extensive use of that formalism for the description of the continuous part of *hybrid* symmetries which we will establish in the following.

3.2 Dynamical $\mathscr{T}$-Systems and $\mathscr{T}$-Symmetries

The fundamental idea that is worked out below is to consider equivariant dynamical systems which are abstractly interconnected by means of an underlying transition graph $\mathscr{T} = (\Lambda, \mathcal{E})$. In doing so, we capture the main part of the hybrid structure a hybrid dynamical system $\mathscr{H}$ possesses, namely the interconnection of graphs and dynamical systems. Even though on this coarse level of description things seem to be extremely similar to coupled cell systems, they eventually are not: The crucial difference lies in the fact that in case of coupled cell systems, one is concerned with a high-dimensional but highly structured dynamical system whose dynamics are influenced by all the system's constituents *at once* while in the distinguished case of hybrid dynamical systems the constituents form the dynamics in a *well-organized time-delayed* manner orchestrated by the structure of the transition graph, the guards and the resets.

It is this primary step which lays the foundations for the hybrid character of hybrid symmetries via the interlocking of discrete graph and continuous dynamical system symmetries. First of all, we formalize the objects to be considered in the following definition keeping in mind that basically from the viewpoint of hybrid dynamical systems we step backwards from Definition 2.1.1. This could also be interpreted as an information deficiency with respect to guards and resets. In fact, this is also equivalent to assuming that guards and resets are trivial, i.e. each guard $\mathcal{G}(e)$ covers the whole phase space $\mathcal{D}(\mathfrak{s}e)$ and the resets essentially come as identities when they are considered in the form of maps. What is still special about the following definition is that we prepare the ground for the incidence of symmetries by providing the spaces with group actions.

3.2.1 Definition (Dynamical $\mathscr{T}$-System). An n-dimensional *dynamical $\mathscr{T}$-system* $\Psi_{\mathscr{T}}$ is a triple $(\mathscr{T}, \Theta, \mathcal{F})$ consisting of

- a directed graph $\mathscr{T} = (\Lambda, \mathcal{E})$,
- a collection $\Theta = \{(D_\lambda, \Phi_{G_\lambda})\}_{\lambda\in\Lambda}$ of phase spaces $D_\lambda \subset \{\lambda\}\times\mathbb{R}^n$ equipped with group actions $\Phi_{G_\lambda} : G_\lambda \times D_\lambda \to D_\lambda$
- and a family $\mathcal{F} = \{f_\lambda : D_\lambda \to \mathbb{R}^n\}_{\lambda\in\Lambda}$ of *vector fields.* $\diamondsuit$

Apparently, this definition needs to be commented on. Firstly, for each $\lambda \in \Lambda$, the pair $\Psi_\lambda = (\Theta(\lambda), f_\lambda)$ sets up a dynamical system by means of the ordinary differential equation $\dot{x} = f_\lambda(x)$ on D_λ. Secondly, note that the assignment of a group action Φ_{G_λ} to each phase space D_λ induces a symmetry group S_λ of the dynamical system $\Psi_\lambda = (\Theta(\lambda), f_\lambda)$ via

$$S_\lambda = \{g_\lambda \in G_\lambda |\ g_\lambda \circ f_\lambda = f_\lambda \circ g_\lambda\} \leq G_\lambda. \tag{3.4}$$

By this means, a dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ can be viewed as a family of equivariant dynamical systems Ψ_λ labeled by the vertices $\lambda \in \Lambda$ of a digraph $\mathscr{T}$. Evidently, the graph structure of $\mathscr{T}$ does not play any role so far. It is now that we take this structure into account: As inspected above, a dynamical $\mathscr{T}$-system *locally* – i.e. for each $\lambda \in \Lambda$ – posesses symmetries in the classical dynamical system's sense, on the one hand. On the other hand, graphs come along with a symmetry concept of their own: The graph $\mathscr{T}$ is accompanied by its automorphism group $\mathrm{Aut}(\mathscr{T})$ which acts on its vertices Λ via

$$\mathrm{Aut}(\mathscr{T}) \times \Lambda \to \Lambda, \quad (\pi, \lambda) \mapsto \pi^{-1}(\lambda), \tag{3.5}$$

meanwhile preserving adjacency.

From a structural point of view, it is legitimate, natural and even necessary to ask for the *symmetry* of a dynamical $\mathscr{T}$-system as an entire mathematical object being a combination of dynamical systems and graphs. Again for clarity, it should be pointed out that a dynamical $\mathscr{T}$-system is certainly qualified as a network of dynamical systems in the truest sense of the word, but still substantially differs from what is called a *coupled cell system* since – unlike coupled cell systems – dynamical $\mathscr{T}$-systems do *not* a priori possess the structure of a (classical) dynamical system, not even of an obviously related map for which equivariance properties could be studied straightforwardly. The key distinction keeping dynamical $\mathscr{T}$-systems from being usual is effectively rooted in the deficiency of information with respect to the graph structure, especially to the meaning of the edges $\mathcal{E}$. However, when interpreting them as couplings between systems by the introduction of a coupling function that connects source and target systems of the respective edges, we do indeed enter the classical framework and for the treatment of the overall symmetry properties the articles [DGS96a] and [DGS96b] apply. From this point of view, dynamical $\mathscr{T}$-systems appear to generalize coupled cell systems and so take on the role of *prototypical* dynamical system networks already accounting for possible symmetries.

In order to grasp the overall symmetry structure of a dynamical $\mathscr{T}$-system, several notational provisions have to be made in advance. Consider a dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$. For each $\lambda \in \Lambda$ there is a group G_λ involved and for an element $g_\lambda \in G_\lambda$ we set

$${}^{g_\lambda}\Psi_\lambda = \big(\Theta(\lambda), g_\lambda^{-1} \circ \mathcal{F}(\lambda)\big) \quad \text{and} \quad \Psi_\lambda^{g_\lambda} = \big(\Theta(\lambda), \mathcal{F}(\lambda) \circ g_\lambda^{-1}\big), \tag{3.6}$$

and – doing that for all $\lambda \in \Lambda$ simultaneously – for $g \in \prod_{\lambda\in\Lambda} G_\lambda$, we write

$$ {}^{g}\Psi_{\mathscr{T}} = \{{}^{g_\lambda}\Psi_\lambda\}_{\lambda\in\Lambda} \quad \text{and} \quad \Psi_{\mathscr{T}}^{g} = \{\Psi_\lambda^{g_\lambda}\}_{\lambda\in\Lambda} \,. \tag{3.7} $$

It is of importance here that a *family* $\{\cdot_\lambda\}_{\lambda\in\Lambda}$ should not be considered to be just a set; it should be rather conceived as a map defined on the index set Λ. Recall that the symmetries of the dynamical system Ψ_λ are encoded by the group

$$ S_\lambda = \{g_\lambda \in G_\lambda \mid {}^{g_\lambda}\Psi_\lambda = \Psi_\lambda^{g_\lambda} \;\} \tag{3.8} $$

and that the vector field $\mathcal{F}(\lambda)$ is S_λ-equivariant. We collect all groups G_λ from Θ and consider the direct products

$$ H = \mathrm{Aut}(\mathscr{T}) \times \prod_{\lambda\in\Lambda} G_\lambda \quad \text{as well as} \quad H_\lambda = \mathrm{Aut}(\mathscr{T}) \times G_\lambda, \quad \lambda \in \Lambda. \tag{3.9} $$

For $\lambda \in \Lambda$ and an element $(\pi, g_\lambda) \in H_\lambda$, we define

$$ (\pi, g_\lambda)\Psi_\lambda = \Big(\Theta(\pi^{-1}(\lambda)), \mathcal{F}(\pi^{-1}(\lambda)) \circ g_{\pi^{-1}(\lambda)}^{-1}\Big) \overset{(3.6)}{=} \Psi_{\pi^{-1}(\lambda)}^{g_{\pi^{-1}(\lambda)}} . \tag{3.10} $$

Accordingly, in the spirit of (3.7), for the dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ and $(\pi, g) \in H$, we set

$$ (\pi, g)\Psi_{\mathscr{T}} = \{(\pi, g_\lambda)\Psi_\lambda\}_{\lambda\in\Lambda} \overset{(3.10)}{=} \Big\{\Psi_{\pi^{-1}(\lambda)}^{g_{\pi^{-1}(\lambda)}}\Big\}_{\lambda\in\Lambda} . \tag{3.11} $$

Looking back, we note that Equations (3.10) and (3.11) represent the first step towards the intertwining of discrete graph and classical symmetries. In order to access the respective data, we utilize the projections introduced below.

3.2.2 Notation. In the following, we denote by $\widehat{\cdot\;\;}$ and $\widehat{\;\;\cdot}$ the projections

$$ H \to \mathrm{Aut}\,(\mathscr{T}) \quad \text{and} \quad H \to \textstyle\prod_{\lambda\in\Lambda} G_\lambda, $$

respectively. Moreover, by $\widehat{\;\;\cdot\;\;}_\lambda$ we denote the projection $H \to G_\lambda$. By abuse of notation, $\widehat{\cdot\;\;}$ and $\widehat{\;\;\cdot}$ as well as $\widehat{\;\;\cdot\;\;}_\lambda$ will also be used to denote the restrictions to subgroups $K \leq H$. Note that these projections and their restrictions to subgroups are group homomorphisms. $\diamondsuit$

When acting on the vertices $\lambda \in \Lambda$ of the graph, a graph automorphism $\pi \in \mathrm{Aut}(\mathscr{T})$ with $G_{\pi^{-1}(\lambda)} = G_\lambda$ for all $\lambda \in \Lambda$ simultaneously acts on the product $\prod_{\lambda\in\Lambda} G_\lambda = \prod_{\lambda\in\Lambda} G_{\pi^{-1}(\lambda)}$; in fact this action is the actual connection of graph

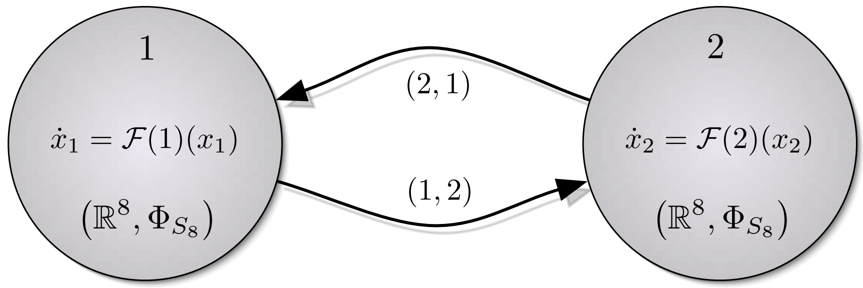


Figure 3.1: Example of a simple dynamical $\mathscr{T}$-system which is strongly related to the hybrid dynamical system introduced in Example 2.1.3

and system symmetries. More precisely, the graph automorphism $\pi \in \mathrm{Aut}(\mathscr{T})$ of $\mathscr{T}$ induces an automorphism of groups via

$$\pi^* : \prod_{\lambda \in \Lambda} G_\lambda \to \prod_{\lambda \in \Lambda} G_\lambda, \quad (\pi^* g)_\lambda = g_{\pi^{-1}(\lambda)}, \tag{3.13}$$

that is to say that the action of $\mathrm{Aut}(\mathscr{T})$ on Λ lifts to an action on the direct group product $\prod_{\lambda \in \Lambda} G_\lambda$. Note that * is *contravariant*, meaning that $(\pi_1 \pi_2)^* = \pi_2^* \circ \pi_1^*$ for two group elements $\pi_1, \pi_2 \in \mathrm{Aut}\,(\mathscr{T})$.

We take a closer look at Example 2.1.3 reducing the hybrid dynamical system $\mathscr{H}$ to a dynamical $\mathscr{T}$-system in order to motivate the notion of $\mathscr{T}$-symmetries.

3.2.3 Example. We consider the eight-dimensional dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ with the transition graph $\mathscr{T}$ given by the vertices $\Lambda = \{1, 2\}$ and the edges $\mathcal{E} = \{(1,2), (2,1)\}$ and phase spaces $\mathcal{D}(1) = \mathcal{D}(2) = \mathbb{R}^8$ equipped with the group action

$$\Phi_{S_8} : S_8 \times \mathbb{R}^8 \to \mathbb{R}^8, \quad (g, x) \mapsto gx$$

with

$$gx = \left(x_{g^{-1}(1)}, x_{g^{-1}(2)}, x_{g^{-1}(3)}, x_{g^{-1}(4)}, x_{g^{-1}(5)}, x_{g^{-1}(6)}, x_{g^{-1}(7)}, x_{g^{-1}(8)}\right)^T \tag{3.14}$$

and S_8 denoting the group of permutations on 8 elements. Consider Fig. 3.1 for a structural visualisation of the system. Let the vector fields $\mathcal{F}(1)$ and

$\mathcal{F}(2)$ be as in Eq. (2.4) and (2.5), respectively. For $g = (15)(26)(37)(48) \in S^8$, we have

$$g\mathcal{F}(1)(x) = \begin{pmatrix} \zeta x_3 - (0.39 + 2\zeta)x_5 - 0.4x_6 + \zeta x_7 + \epsilon x_5 x_6^2 \\ \zeta x_4 + 0.04x_5 - (0.39 + 2\zeta)x_6 + \zeta x_8 + 2.5\epsilon x_5 x_6 \\ \zeta x_5 - (0.39 + \zeta)x_7 - 0.4x_8 + \epsilon x_7 x_8^2 \\ \zeta x_6 + 0.04x_7 - (0.39 + \zeta)x_8 + 2.5\epsilon x_7 x_8 \\ -(0.39 + \zeta)x_1 - 0.4x_2 + \zeta x_3 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39 + \zeta)x_2 + \zeta x_4 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39 + 2\zeta)x_3 - 0.4x_4 + \zeta x_5 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39 + 2\zeta)x_4 + \zeta x_6 + 2.5\epsilon x_3 x_4 \end{pmatrix} \tag{3.15}$$

and

$$\mathcal{F}(2)(gx) = \begin{pmatrix} -(0.39 + 2\zeta)x_5 - 0.4x_6 + \zeta x_7 + \zeta x_3 + \epsilon x_5 x_6^2 \\ 0.04x_5 - (0.39 + 2\zeta)x_6 + \zeta x_8 + \zeta x_4 + 2.5\epsilon x_5 x_6 \\ \zeta x_5 - (0.39 + \zeta)x_7 - 0.4x_8 + \epsilon x_7 x_8^2 \\ \zeta x_6 + 0.04x_7 - (0.39 + \zeta)x_8 + 2.5\epsilon x_7 x_8 \\ -(0.39 + \zeta)x_1 - 0.4x_2 + \zeta x_3 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39 + \zeta)x_2 + \zeta x_4 + 2.5\epsilon x_1 x_2 \\ \zeta x_5 + \zeta x_1 - (0.39 + 2\zeta)x_3 - 0.4x_4 + \epsilon x_3 x_4^2 \\ \zeta x_6 + \zeta x_2 + 0.04x_3 - (0.39 + 2\zeta)x_4 + 2.5\epsilon x_3 x_4 \end{pmatrix}. \tag{3.16}$$

Comparing Eqs. (3.15) and (3.16), we find that

$$g \circ \mathcal{F}(1) = \mathcal{F}(2) \circ g \quad \text{as well as} \quad \mathcal{F}(1) \circ g = g \circ \mathcal{F}(2) \tag{3.17}$$

since g is self-inverse. Observe that the unique non-trivial graph automorphism of $\mathscr{T}$ is given by

$$\pi : \Lambda \to \Lambda, \quad \pi(1) = 2,\ \pi(2) = 1 \tag{3.18}$$

and together with Eq. (3.17) we obtain

$$\mathcal{F}(\pi^{-1}(1)) \circ g = g \circ \mathcal{F}(1) \quad \text{and} \quad \mathcal{F}(\pi^{-1}(2)) \circ g = g \circ \mathcal{F}(2). \tag{3.19}$$

With $g_1 = g_2 = g$ we can write

$$\mathcal{F}(\pi^{-1}(\lambda)) \circ g_\lambda = g_\lambda \circ \mathcal{F}(\lambda) \tag{3.20}$$

for all $\lambda \in \Lambda$. $\diamondsuit$

We now turn to the formal definition of *$\mathscr{T}$-symmetries* which are designed to capture the content of symmetry exhibited by dynamical $\mathscr{T}$-systems.

3.2.4 Definition ($\mathscr{T}$-Symmetry). Let $\Psi_{\mathscr{T}}$ be a dynamical $\mathscr{T}$-system. An element $(\pi, g) \in H$ is a *pre-$\mathscr{T}$-symmetry* if the identity

$$(\pi, g)\Psi_{\mathscr{T}} = {}^{g}\Psi_{\mathscr{T}} \tag{3.21}$$

holds. The collection of all pre-$\mathscr{T}$-symmetries is denoted by $\mathfrak{S}_\flat$. An element (π, g) of $\mathfrak{S}_\flat$ is a *$\mathscr{T}$-symmetry* if the additional constancy condition

$$\pi^* g = g \qquad \text{for all } \pi \in \widehat{\mathfrak{S}}_\flat \tag{3.22}$$

is fulfilled. The collection of all $\mathscr{T}$-symmetries is denoted by $\mathfrak{S}$. $\diamondsuit$

The essence of Eq. (3.21) consists in the very original conception that the application of a symmetry leaves things unchanged. More specifically, in this special regime of dynamical $\mathscr{T}$-systems as I introduced them above, the graph automorphism π shuffles the graph $\mathscr{T}$ via permutation of the vertices leaving it invariant; meanwhile, the accumulated symmetry $g \in \prod_{\lambda\in\Lambda} G_\lambda$ transforms the nodal dynamical systems in just the correct way such that on the global scale the $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ is preserved by (π, g).

However, in this context Eq. (3.22) may appear unusual and surprising. This constancy or fixed-point condition has to be considered for purely algebraic reasons. In the treatment of every heterogeneously composite system one has to think about consistency or composability at some point, at least whenever one is to deduce global statements out of local ones or even at the beginning in the phase of modeling. It is the same thing here: The construction of a global symmetry notion for $\mathscr{T}$-systems including algebraic structure (in our case of groups) requires an assumption ensuring that all supporting components fit together in an algebraic sense. This is exactly what Eq. (3.22) is about.

We characterize $\mathscr{T}$-symmetries in terms of phase spaces and symmetry properties of the vector fields.

3.2.5 Lemma (Characterization of (Pre-)$\mathscr{T}$-Symmetries). *Let $\Psi_{\mathscr{T}}$ be a dynamical $\mathscr{T}$-system. An element $(\pi, g) \in H$ is a pre-$\mathscr{T}$-symmetry of $\Psi_{\mathscr{T}}$ if and only if*

$$\Theta(\pi^{-1}(\lambda)) = \Theta(\lambda) \quad \textit{and} \quad \mathcal{F}(\pi^{-1}(\lambda)) \circ g^{-1}_{\pi^{-1}(\lambda)} = g^{-1}_\lambda \circ \mathcal{F}(\lambda) \tag{3.23}$$

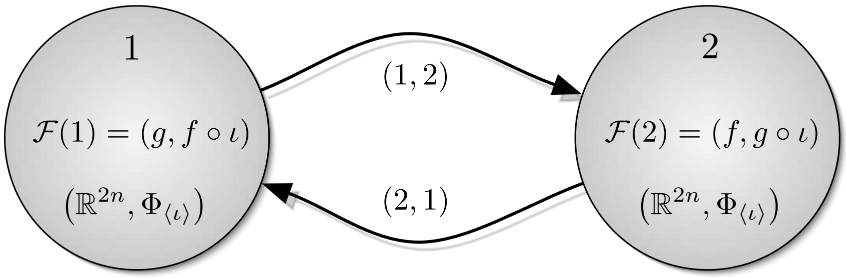
for all $\lambda \in \Lambda$.

Proof. Just unwind Definition 3.2.4 by comparison of Equations (3.10) and (3.21). ∎

Note that $\Theta(\pi^{-1}(\lambda)) = \Theta(\lambda)$ implies that the corresponding groups and their actions coincide, more precisely $G_{\pi^{-1}(\lambda)} = G_\lambda$ and $\Phi_{G_{\pi^{-1}(\lambda)}} = \Phi_{G_\lambda}$ for all $\lambda \in \Lambda$.

Before we turn to algebraic considerations, we again examine an example to let things appear more clearly.

3.2.6 Example. Consider the general dynamical $\mathscr{T}$-system as displayed in Figure 3.2. Note that the coordinate-changing homeomorphism ι is an involution and thus generates the group $\langle\iota\rangle \cong \mathbb{Z}_2$, which acts on $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. The vector fields $\mathcal{F}(1)$ and $\mathcal{F}(2)$ are topologically conjugated via ι:



$$f, g : \mathbb{R}^{2n} \to \mathbb{R}^n$$

$$\iota : \mathbb{R}^{2n} \xrightarrow{\approx} \mathbb{R}^{2n}, \quad \iota(x_1, x_2) = (x_2, x_1)$$

FIGURE 3.2: Prototypical dynamical $\mathscr{T}$-system exhibiting $\mathscr{T}$-symmetries

$$(\mathcal{F}(1) \circ \iota)(x_1, x_2) = \begin{pmatrix} g(x_2, x_1) \\ f(x_1, x_2) \end{pmatrix} = (\iota \circ \mathcal{F}(2))(x_1, x_2).$$

For further investigation, we need to draw a distinction between the cases $f = g$ and $f \neq g$. First, we assume $f \neq g$. If so, neither $\mathcal{F}(1)$ nor $\mathcal{F}(2)$ is equivariant with respect to ι. Figure 3.2 reveals $\operatorname{Aut}(\mathscr{T}) = \langle(12)\rangle = S_2 \cong \mathbb{Z}_2$ and $(\pi, g_1, g_2) \in S_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is a $\mathscr{T}$-symmetry of the system if and only if

$$(\pi, g_1, g_2) = 1_H \qquad \text{or} \qquad (\pi, g_1, g_2) = ((12), \iota, \iota)\,,$$

which implies $\mathfrak{S} = \langle((12), \iota, \iota)\rangle \cong \mathbb{Z}_2$. In case $f = g$ one has $\mathcal{F}(1) = \mathcal{F}(2) = F$ and $\mathbb{Z}_2$-equivariance of F, thus yielding $\mathscr{T}$-symmetry $\mathfrak{S} \cong S_2 \times \mathbb{Z}_2$.

Note that – away from the distinction of the cases $f = g$ and $f \neq g$ – the above statements hold for any pair of vector fields f and g and are therefore completely independent from the explicit choice of these vector fields.

In the light of these findings, we reconsider Example 3.2.3. A closer inspection of the vector fields $\mathcal{F}(1)$ and $\mathcal{F}(2)$ (cp. Eqs. (2.4) and (2.5)) shows that we have

$$\mathcal{F}(1)(x) = \begin{pmatrix} g(y_1, y_2) \\ f(y_2, y_1) \end{pmatrix} \quad \text{and} \quad \mathcal{F}(2)(x) = \begin{pmatrix} f(y_1, y_2) \\ g(y_2, y_1) \end{pmatrix} \tag{3.26}$$

using the maps

$$f(x_1, \ldots, x_8) = \begin{pmatrix} -(0.39 + 2\zeta)x_1 - 0.4x_2 + \zeta x_3 + \zeta x_7 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39 + 2\zeta)x_2 + \zeta x_4 + \zeta x_8 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39 + \zeta)x_3 - 0.4x_4 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39 + \zeta)x_4 + 2.5\epsilon x_3 x_4 \end{pmatrix} \tag{3.27}$$

and

$$g(x_1, \ldots, x_8) = \begin{pmatrix} -(0.39 + \zeta)x_1 - 0.4x_2 + \zeta x_3 + \epsilon x_1 x_2^2 \\ 0.04x_1 - (0.39 + \zeta)x_2 + \zeta x_4 + 2.5\epsilon x_1 x_2 \\ \zeta x_1 - (0.39 + 2\zeta)x_3 - 0.4x_4 + \zeta x_5 + \epsilon x_3 x_4^2 \\ \zeta x_2 + 0.04x_3 - (0.39 + 2\zeta)x_4 + \zeta x_6 + 2.5\epsilon x_3 x_4 \end{pmatrix} \tag{3.28}$$

and the grouping

$$y_1 = (x_1, x_2, x_3, x_4), \;\; y_2 = (x_5, x_6, x_7, x_8) \in \mathbb{R}^4. \tag{3.29}$$

In terms of y_1 and y_2, we can write f in the following form:

$$f(y_1, y_2) = (A_1 \;\, A_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + h(y_1) \tag{3.30}$$

with

$$A_1 = \begin{pmatrix} -(0.39 + 2\zeta) & -0.4 & \zeta & 0 \\ 0.04 & -(0.39 + 2\zeta) & 0 & \zeta \\ \zeta & 0 & -(0.39 + \zeta) & -0.4 \\ 0 & \zeta & 0.04 & -(0.39 + \zeta) \end{pmatrix}, \tag{3.31}$$

$$A_2 = \begin{pmatrix} 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

and

$$h : \mathbb{R}^4 \to \mathbb{R}^4, \quad h(x_1, x_2, x_3, x_4) = \epsilon \begin{pmatrix} x_1 x_2^2 \\ 2.5 x_1 x_2 \\ x_3 x_4^2 \\ 2.5 x_3 x_4 \end{pmatrix}. \tag{3.33}$$

Similarly, for g we obtain

$$g(y_1, y_2) = (B_1 \ B_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + h(y_1), \tag{3.34}$$

where the matrices B_1 and B_2 are given by

$$B_1 = \begin{pmatrix} -(0.39 + \zeta) & -0.4 & \zeta & 0 \\ 0.04 & -(0.39 + \zeta) & 0 & \zeta \\ \zeta & 0 & -(0.39 + 2\zeta) & -0.4 \\ 0 & \zeta & 0.04 & -(0.39 + 2\zeta) \end{pmatrix} \tag{3.35}$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \end{pmatrix} = A_2^T. \tag{3.36}$$

Thus, the dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ discussed in Example 3.2.3 exactly fits into the class of systems illustrated by Fig. 3.2 with $n = 4$ and f, g as above. Notably, $f \neq g$. Moreover, the symmetry element $g = (15)(26)(37)(48) \in S_8$ figured out in Example 3.2.3 translates to the involution ι on $\mathbb{R}^4 \times \mathbb{R}^4$ when we take into consideration the special structure of $\mathcal{F}(1)$ and $\mathcal{F}(2)$ as discovered above. Hence, $\Psi_{\mathscr{T}}$ turns out to have *at least* the $\mathscr{T}$-symmetries $\mathfrak{G} = \langle(\pi, \iota, \iota)\rangle \cong \mathbb{Z}_2$. In fact, the vector fields $\mathcal{F}(1)$ and $\mathcal{F}(2)$ exhibit even richer structure which we have not discussed so far and which gives rise to additional $\mathscr{T}$-symmetries. We will track the remaining symmetries later in Example 3.2.14. $\diamondsuit$

The fact that $\mathfrak{S}$ turns out to possess the structure of a group in Example 3.2.6 is no accident. We formally treat this observation in the following.

3.2.7 Proposition (Algebraic Structure of $\mathscr{T}$-Symmetries). *The $\mathscr{T}$-symmetries $\mathfrak{S}$ of a dynamical $\mathscr{T}$-system form a group.*

Proof. We verify the group axioms: The element $1_H = (\mathrm{id}_\Lambda, 1_{\widehat{H}}) \in H = \mathrm{Aut}(\mathscr{T}) \times \prod_{\lambda\in\Lambda} G_\lambda$ is easily seen to play the role of the neutral element since

$$1_H \Psi_{\mathscr{T}} = \Psi_{\mathscr{T}} = {}^{1_{\widehat{H}}}\Psi_{\mathscr{T}}.$$

Associativity is verified via a straightforwardly executed series of elementary computations using Definition 3.2.4 or Lemma 3.2.5. For two $\mathscr{T}$-symmetries $(\pi_1, g_1), (\pi_2, g_2) \in \mathfrak{S}$, we compute $(\pi_2, g_2)(\pi_1, g_1) = (\pi_2\pi_1, g_2g_1)$ and

$$\begin{aligned}
&(\pi_2\pi_1, g_2g_1)\,\Psi_\lambda \\
\overset{(3.10)}{=}\; & \left(\Theta\left(\pi_1^{-1}(\pi_2^{-1}(\lambda))\right), \mathcal{F}\left(\pi_1^{-1}(\pi_2^{-1}(\lambda))\right) \circ (g_1)^{-1}_{\pi_1^{-1}(\pi_2^{-1}(\lambda))} \circ (g_2)^{-1}_{\pi_1^{-1}(\pi_2^{-1}(\lambda))}\right) \\
\overset{(3.23),\ \pi_2^* g_2 = g_2}{=}\; & \left(\Theta\left(\pi_2^{-1}(\lambda)\right), (g_1)^{-1}_{\pi_2^{-1}(\lambda)} \circ \mathcal{F}\left(\pi_2^{-1}(\lambda)\right) \circ (g_2)^{-1}_{\pi_2^{-1}(\lambda)}\right) \\
\overset{\pi_2^* g_1 = g_1,\ (3.23)}{=}\; & \left(\Theta(\lambda), (g_1)^{-1}_\lambda \circ (g_2)^{-1}_\lambda \circ \mathcal{F}(\lambda)\right) \\
=\; & \left(\Theta(\lambda), (g_2g_1)^{-1}_\lambda \circ \mathcal{F}(\lambda)\right) \\
\overset{(3.10)}{=}\; & {}^{(g_2g_1)_\lambda}\Psi_\lambda
\end{aligned}$$

for all $\lambda \in \Lambda$, implying $(\pi_2, g_2)(\pi_1, g_1) \in \mathfrak{S}$. In order to prove the existence of inverses, set $\widetilde{\lambda} := \pi(\lambda)$ for $(\pi, g) \in \mathfrak{S}$ and $\lambda \in \Lambda$. Observe that

$$\left(\Theta(\lambda), \mathcal{F}(\lambda) \circ g_\lambda^{-1}\right) = \left(\Theta(\widetilde{\lambda}), g_{\widetilde{\lambda}}^{-1} \circ \mathcal{F}(\widetilde{\lambda})\right) \tag{3.37}$$

by means of (3.23) and proceed:

$$\begin{aligned}
(\pi, g)^{-1}\Psi_\lambda \;&\overset{(3.10)}{=}\; \left(\Theta(\widetilde{\lambda}), \mathcal{F}(\widetilde{\lambda}) \circ g_{\widetilde{\lambda}}\right) \\
&\overset{(3.22):\ (\pi^{-1})^* g = g}{=}\; \left(\Theta(\widetilde{\lambda}), \mathcal{F}(\widetilde{\lambda}) \circ g_\lambda\right) \\
&\overset{(3.37),(3.22)}{=}\; (\Theta(\lambda), g_\lambda \circ \mathcal{F}(\lambda)) \\
&=\; {}^{g^{-1}}\Psi_\lambda.
\end{aligned}$$

Hence, $(\pi, g)^{-1} \in \mathfrak{S}$ and, consequently, $\mathfrak{S}$ is found to be a group. ∎

As mentioned before, the automorphism group $\mathrm{Aut}\,(\mathscr{T})$ acts on $\widehat{H} = \prod_{\lambda\in\Lambda} G_\lambda$ via the group homomorphism

$$^* : \mathrm{Aut}\,(\mathscr{T}) \to \mathrm{Aut}(\widehat{H}), \quad \pi \mapsto \pi^*. \tag{3.38}$$

For a subgroup $\Gamma \leq \mathrm{Aut}\,(\mathscr{T})$, the fixed-point space of Γ in $\widehat{H}$ with respect to (3.38) is given by

$$\mathrm{Fix}_{\widehat{H}}(\Gamma) := \left\{g \in \widehat{H} \;\middle|\; \pi^* g = g \text{ for all } \pi \in \Gamma\right\}. \tag{3.39}$$

Note that $\mathrm{Fix}_{\widehat{H}}(\Gamma)$ inherits the structure of a group and thus is a subgroup of $\widehat{H}$. This provides us with the following structural result.

3.2.8 Proposition. *For the $\mathscr{T}$-symmetry group $\mathfrak{S}$ of a dynamical $\mathscr{T}$-system, one has*

$$\mathfrak{S} \leq \mathrm{Aut}\,(\mathscr{T}) \times \mathrm{Fix}_{\widehat{H}}(\widehat{\mathfrak{S}}). \tag{3.40}$$

In order to further characterize the group of $\mathscr{T}$-symmetries in terms of phase spaces and vector fields, we introduce the Θ- and $\mathcal{F}$-*coincidence sets* in $\lambda \in \Lambda$ by

$$C_{\Theta,\lambda} := \left\{\pi \in \mathrm{Aut}\,(\mathscr{T}) \;\middle|\; \Theta(\pi^{-1}(\lambda)) = \Theta(\lambda)\right\}, \tag{3.41}$$

and

$$C_{\mathcal{F},\lambda} := \left\{\pi \in \mathrm{Aut}\,(\mathscr{T}) \;\middle|\; \mathcal{F}(\pi^{-1}(\lambda)) = \mathcal{F}(\lambda)\right\}, \tag{3.42}$$

analogously. Moreover, for the dynamical system $\Psi_\lambda = (\Theta(\lambda), \mathcal{F}(\lambda))$, we set

$$C_{\Psi,\lambda} := C_{\Theta,\lambda} \cap C_{\mathcal{F},\lambda}. \tag{3.43}$$

Finally, based on the Θ-coincidence set $C_{\Theta,\lambda}$, we define the *$\mathcal{F}$-similarity or -conjugacy set*

$$C^{\sim}_{\mathcal{F},\lambda} := \left\{\pi \in C_{\Theta,\lambda} \;\middle|\; \exists\, g_\lambda \in G_\lambda : \mathcal{F}(\pi^{-1}(\lambda)) = g_\lambda^{-1} \circ \mathcal{F}(\lambda) \circ g_\lambda\right\}. \tag{3.44}$$

Based on Lemma 3.2.5, we figure out the characteristics of $\mathscr{T}$-symmetries using coincidence and similarity sets. This contributes to the understanding of the (global) relationship between a $\mathscr{T}$-system and its $\mathscr{T}$-symmetries.

3.2.9 Proposition. *Let $\Psi_{\mathscr{T}}$ be a dynamical $\mathscr{T}$-system with $\mathscr{T}$-symmetry group $\mathfrak{S}$. Then the phase spaces $\Theta(\lambda)$ coincide along $\widehat{\mathfrak{S}}$-orbits, more precisely:*

$$C_{\Theta,\lambda} = \widehat{\mathfrak{S}} \qquad \text{for all } \lambda \in \Lambda. \tag{3.45}$$

Moreover, for all $(\pi, g) \in \mathfrak{S}$*, the vector fields* $\mathcal{F}(\pi^{-1}(\lambda))$ *and* $\mathcal{F}(\lambda)$ *are topologically conjugated, i. e.*

$$\mathcal{F}(\pi^{-1}(\lambda)) = g_\lambda^{-1} \circ \mathcal{F}(\lambda) \circ g_\lambda, \tag{3.46}$$

or, equivalently,

$$C^{\sim}_{\mathcal{F},\lambda} = \widehat{\mathfrak{S}} \qquad \textit{for all} \;\; \lambda \in \Lambda. \tag{3.47}$$

In particular, both $C_{\Theta,\lambda}$ *and* $C^{\sim}_{\mathcal{F},\lambda}$ *are subgroups of* $\mathrm{Aut}\,(\mathscr{T})$*. Furthermore, if the vector fields coincide along the orbits of* $\widehat{\mathfrak{S}}$ *meaning that* $C_{\Psi,\lambda} = \widehat{\mathfrak{S}}$ *for all* $\lambda \in \Lambda$*, then*

$$\mathfrak{S} \cong \widehat{\mathfrak{S}} \times \widehat{\mathfrak{S}}. \tag{3.48}$$

Proof. First of all, since $\mathfrak{S}$ is a group by Proposition 3.2.7 and the projections $\widehat{\cdot}$ and $\widehat{\cdot}$ are group homomorphisms, $\widehat{\mathfrak{S}}$ and $\widehat{\mathfrak{S}}$ are subgroups of $\widehat{H} = \mathrm{Aut}\,(\mathscr{T})$ and $\widehat{H} = \prod_{\lambda\in\Lambda} G_\lambda$, respectively. Lemma 3.2.5 ensures that for all pre-$\mathscr{T}$-symmetries and - in particular - for all $\mathscr{T}$-symmetries $(\pi, g) \in \mathfrak{S}$ and all $\lambda \in \Lambda$, we have $\Theta(\pi^{-1}(\lambda)) = \Theta(\lambda)$ which is equivalent to $C_{\Theta,\lambda} = \widehat{\mathfrak{S}}$ for all $\lambda \in \Lambda$.

Again by means of Lemma 3.2.5, for a pre-$\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S}_\flat$, we have the relations

$$\mathcal{F}(\pi^{-1}(\lambda)) \circ g^{-1}_{\pi^{-1}(\lambda)} = g_\lambda^{-1} \circ \mathcal{F}(\lambda), \quad \lambda \in \Lambda.$$

Incorporating the constancy condition (3.22) which alternatively reads $g_{\pi^{-1}(\lambda)} = g_\lambda$ for all $\pi \in \widehat{\mathfrak{S}}$ and $\lambda \in \Lambda$, we end up with

$$\mathcal{F}(\pi^{-1}(\lambda)) = g_\lambda^{-1} \circ \mathcal{F}(\lambda) \circ g_\lambda \quad \text{for all} \;\; \lambda \in \Lambda$$

expressing topological conjugacy of the vector fields in virtue of $\mathscr{T}$-symmetries. Since this relation holds for each $\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S}$, we obtain $C^{\sim}_{\mathcal{F},\lambda} = C_{\Theta,\lambda} = \widehat{\mathfrak{S}}$ for the $\mathcal{F}$-conjugacy sets.

For the last statement, it suffices to show that for each $\pi \in \widehat{\mathfrak{S}}$ and each $g \in \widehat{\mathfrak{S}}$, the pair (π, g) is a $\mathscr{T}$-symmetry of $\Psi_{\mathscr{T}}$. By assumption, we have $C_{\Psi,\lambda} = \widehat{\mathfrak{S}}$. For $g \in \widehat{\mathfrak{S}}$ there exists $\widetilde{\pi} \in \widehat{\mathfrak{S}}$ such that $(\widetilde{\pi}, g) \in \mathfrak{S}$. Due to the first part of the proof, we know that the identity $\mathcal{F}(\lambda) \circ g_\lambda = g_\lambda \circ \mathcal{F}(\widetilde{\pi}^{-1}(\lambda))$ holds, and that $\Theta(\pi^{-1}(\lambda)) = \Theta(\lambda)$ is true for all $\pi \in \widehat{\mathfrak{S}}$. The assumption $\Psi_{\pi^{-1}(\lambda)} = \Psi_\lambda$ implies $\mathcal{F}(\pi^{-1}(\lambda)) = \mathcal{F}(\lambda)$ for all $\pi \in \widehat{\mathfrak{S}}$. This leads to the equivariance condition

$\mathcal{F}(\lambda) \circ g_\lambda = g_\lambda \circ \mathcal{F}(\lambda)$ imposed on g, which is obviously independent of any $\pi \in \widehat{\mathfrak{S}}$. Thus, (π, g) is a $\mathscr{T}$-symmetry for every choice of $\pi \in \widehat{\mathfrak{S}}$, which verifies the last statement and thereby completes the proof of the proposition. ■

In the following, we comment on the relationship between classical equivariance of vector fields and the new type of equivariance occuring for $\mathscr{T}$-symmetries.

3.2.10 Remark (Weak Equivariance). As Proposition 3.2.9 tells us, the outcome of the symmetry concept for $\mathscr{T}$-systems (as defined in Definition 3.2.4) is a collection of equations

$$\left\{g_\lambda \circ \mathcal{F}(\pi^{-1}(\lambda)) = \mathcal{F}(\lambda) \circ g_\lambda\right\}_{\lambda \in \Lambda}, \tag{3.49}$$

which I will term *weak equivariance* compared to the (classical) *equivariance* $g \circ F = F \circ g$ for a map F and a group element $g \in G$.

Let me comment on this. Weak equivariance may be considered to be a generalization of classical equivariance in the following sense: Observing that weak equivariance occurs for a *family* $\mathcal{F} = \{\mathcal{F}(\lambda)\}_{\lambda \in \Lambda}$ of maps rather than for an individual map, we see that there are essentially two distinct ways for weak equivariance to collapse to classical equivariance. Firstly, if the graph $\mathscr{T}$ is trivial in the sense that it solely consists of a simple node λ_0 (this corresponds to the case of a simple map), its unique automorphism π is forced to be the identity, namely $\pi^{-1}(\lambda_0) = \lambda_0$, and Eq. (3.49) results in a simple equivariance condition for $\mathcal{F}(\lambda_0)$. Secondly, in case the graph $\mathscr{T}$ is not trivial and the graph automorphism π (stemming from the $\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S}$) fixes some vertex $\lambda_0 \in \Lambda$, then the vector field $\mathcal{F}(\lambda_0)$ experiences equivariance with respect to g_{λ_0}. We put this last observation in the following

3.2.11 Lemma. *Let $\Sigma_\lambda = \{\pi \in \mathrm{Aut}\,(\mathscr{T}) \mid \pi^{-1}(\lambda) = \lambda\}$ be the isotropy group of the discrete state $\lambda \in \Lambda$. Then $H_\lambda \subset S_\lambda$ with*

$$H_\lambda = \left\{ g_\lambda \in G_\lambda \;\middle|\; (\pi, g) \in \mathfrak{S} \cap \left(\Sigma_\lambda \times \prod_{\lambda \in \Lambda} G_\lambda \right) \right\}$$

and S_λ captures the symmetries of the dynamical system Ψ_λ.[1]

[1] A slightly more general version of this statement is contained in the later Proposition

Thus classical equivariance may turn up locally as a special form of weak equivariance.

The other significant aspect arguing for the attribute *weak* concerns the *topological conjugacy* (which has already been mentioned in the formulation of Proposition 3.2.9) of the vector fields $\mathcal{F}(\pi^{-1}(\lambda))$ and $\mathcal{F}(\lambda)$ under the action of a $\mathscr{T}$-symmetry (π, g). Dynamically, this means that the dynamics of the dynamical systems $\Psi_{\pi^{-1}(\lambda)}$ and Ψ_λ are strongly related: The homeomorphism g_λ transforms orbits of $\Psi_{\pi^{-1}(\lambda)}$ into orbits of Ψ_λ. Against this background, the predicate *weak* is meant to refer to this softening of dynamic equality to similarity in the topological sense. $\diamondsuit$

By means of Propositions 3.2.8 and 3.2.9, we are enabled to deduce structural information for the $\mathscr{T}$-symmetry group in some special cases.

3.2.12 Corollary. *Let $\Psi_{\mathscr{T}}$ be a dynamical $\mathscr{T}$-system. If $\mathscr{T}$ is of trivial symmetry or if $\widehat{\mathfrak{S}} \leq \mathrm{Aut}(\mathscr{T})$ is the trivial subgroup, then*

$$\mathfrak{S} \cong \widehat{\mathfrak{S}} = \prod_{\lambda \in \Lambda} S_\lambda, \tag{3.52}$$

where $S_\lambda \leq G_\lambda$ is the symmetry group of the dynamical system Ψ_λ (cp. (3.8)). If $\widehat{\mathfrak{S}}$ acts transitively on Λ and $C_{\mathcal{F},\lambda} = \widehat{\mathfrak{S}}$ for all $\lambda \in \Lambda$, i.e. $F = \mathcal{F}(\lambda)$ for all λ, then

$$\mathfrak{S} \cong \mathrm{Aut}\,(\mathscr{T}) \times S, \tag{3.53}$$

where S denotes the symmetry group with respect to the vector field F.

Proof. In case $\widehat{\mathfrak{S}} = 1$, we detect $\mathfrak{S} \cong \widehat{\mathfrak{S}} \leq \mathrm{Fix}_{\widehat{H}}(\widehat{\mathfrak{S}}) = \prod_{\lambda \in \Lambda} G_\lambda$, using Proposition 3.2.8. Moreover, the identity $(\pi, g)\Psi_{\mathscr{T}} = {}^g\Psi_{\mathscr{T}}$ simplifies to the

3.2.13. Note that with the subgroup

$$H_{\pi,\lambda} = \widehat{(H_\pi)}_\lambda, \quad \text{where} \quad H_\pi = \widehat{\cdot}^{-1}(\pi) \cap \mathfrak{S} \tag{3.50}$$

the set $H_\lambda = \{g_\lambda \in G_\lambda \mid (\pi, g) \in \mathfrak{S} \cap (\Sigma_\lambda \times \prod_{\lambda \in \Lambda} G_\lambda)\}$ can be written in the form

$$H_\lambda = \bigcup_{\pi \in \Sigma_\lambda} H_{\pi,\lambda} \subset G_\lambda. \tag{3.51}$$

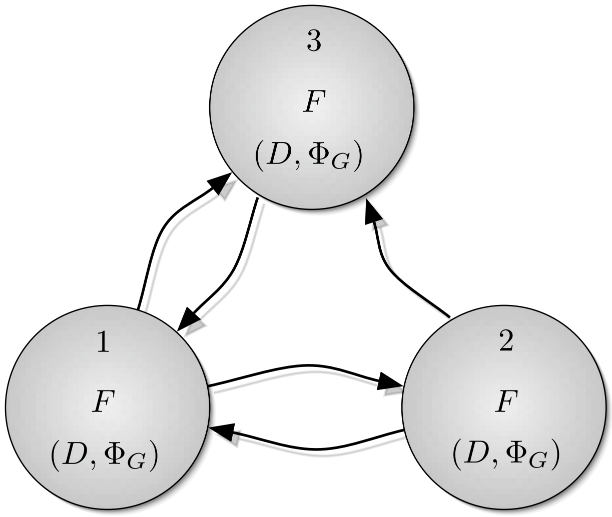
As a union of subgroups, H_λ does not necessarily show the structure of a group itself.

equivariance condition $\mathcal{F}(\lambda) \circ g_\lambda = g_\lambda \circ \mathcal{F}(\lambda)$ for every $\lambda \in \Lambda$ and forces $g \in \prod_{\lambda\in\Lambda} S_\lambda$. Thus, $\widehat{\mathfrak{S}} = \prod_{\lambda\in\Lambda} S_\lambda$.

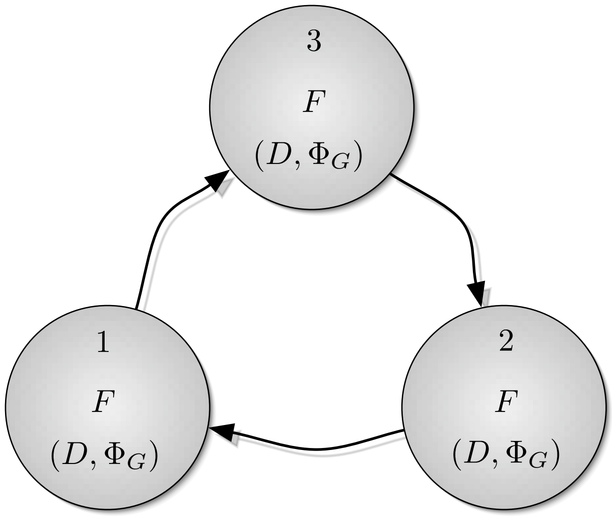
In case, $\widehat{\mathfrak{S}}$ acts transitively on Λ, the coincidence $C_{\Psi,\lambda} = \widehat{\mathfrak{S}}$ of dynamical systems on $\widehat{\mathfrak{S}}$-orbits ensures that $\Psi_\lambda = \Psi$ is constant on Λ. This implies $\mathrm{Fix}_{\widehat{H}}(\widehat{\mathfrak{S}}) = \Delta(G^{\mathrm{card}(\Lambda)}) \cong G$, where G denotes the group acting on each $\mathcal{D}(\lambda)$ and $\Delta(G^n)$ is the diagonal subgroup of G^n. Using Proposition 3.2.8, we find $\widehat{\mathfrak{S}} \leq G$. Again, due to the constancy $\mathcal{F}(\lambda) = F$, the $\mathscr{T}$-symmetry condition $(\pi, g)\Psi_{\mathscr{T}} = {}^g\Psi_{\mathscr{T}}$ breaks down to classical equivariance yielding $\widehat{\mathfrak{S}} \cong S$, where $S \leq G$ denotes the group of symmetries corresponding to Ψ. Furthermore, for the discrete part of $\mathfrak{S}$, we find $\widehat{\mathfrak{S}} = \mathrm{Aut}\,(\mathscr{T})$. Finally, Proposition 3.2.9 gives $\mathfrak{S} = \widehat{\mathfrak{S}} \times \widehat{\mathfrak{S}}$, which leaves us to conclude that $\mathfrak{S} \cong \mathrm{Aut}\,(\mathscr{T}) \times S$. ■

For an illustration of Corollary 3.2.12, see Figure 3.3 which shows two examples of equivariant dynamical $\mathscr{T}$-systems with identical local dynamics ruled by Ψ_λ and G-equivariant vector fields. In particular, this example shows the interaction of discrete graph symmetries and continuous dynamical system symmetries as the transition graph changes symmetry. What is more, we see that the interaction of discrete graph symmetries and local dynamical systems' symmetries is of a qualitatively very noteworthy kind: In case of their presence, the graph symmetries restrain the local symmetries, and the more symmetric the transition graph happens to be the stronger the local symmetries are forced to settle on diagonals and therefore are sentenced to contraction. To put it in an even more vivid way, the comparatively loose direct product $\prod_{\lambda\in\Lambda} G_\lambda$ is stamped on a strict polydiagonal structure by the graph symmetries. Figure 3.4 provides an exemplary illustration of this fact.

Beside Figures 3.3 and 3.4 another prototypical instance that considerably contributes to the understanding of the nature of $\mathscr{T}$-symmetries is discussed in Example 3.2.6. It demonstrates the generation of mixed-type symmetries in a situation where both vector fields are neither identical nor exhibit any symmetry. This almost suggests asking for details in a situation where the vector fields still do not coincide, but both bring into equation non-trivial symmetry properties. More precisely, we ask the following question: In which manner are symmetries of one local system linked with symmetries of other systems if $\mathscr{T}$-symmetries are present or – more carefully worded – under which conditions and in which form do $\mathscr{T}$-symmetries arise in a situation like that? The follow-

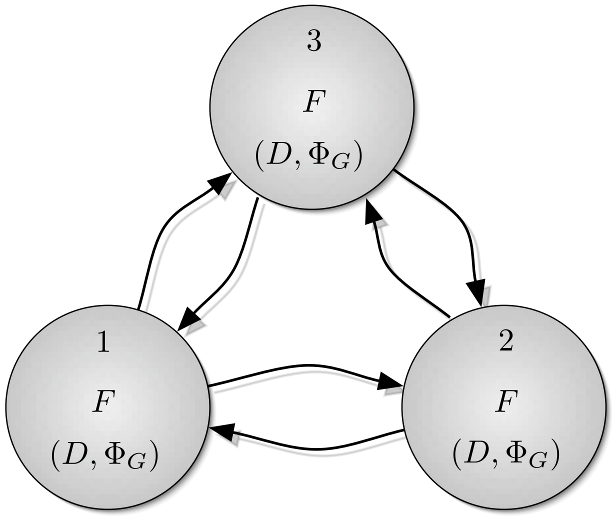


(a)

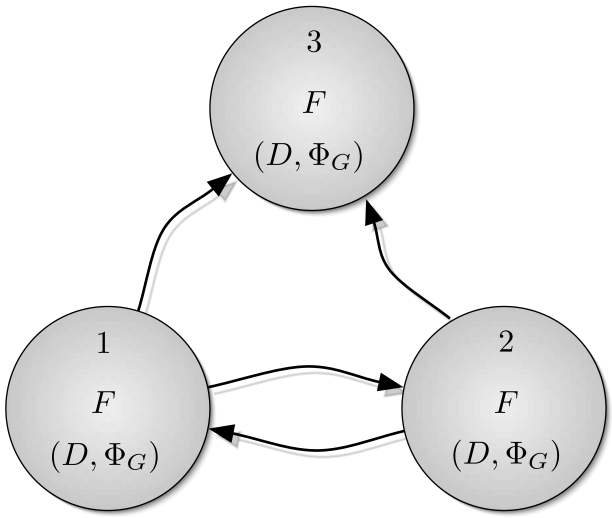


(b)

Figure 3.3: Dynamical $\mathscr{T}$-systems with $\mathscr{T}$-symmetry groups $\mathfrak{S} \cong G \times G \times G$ (trivial transition graph symmetry) and $\mathfrak{S} \cong \mathbb{Z}_3 \times G$ (transitive action of $\widehat{\mathfrak{S}}$)



(a)



(b)

Figure 3.4: Dynamical $\mathscr{T}$-systems with $\mathscr{T}$-symmetry groups $\mathfrak{G} \cong S_3 \times G$ (complete symmetry of $\mathscr{T}$) and $\mathfrak{G} \cong \mathbb{Z}_2 \times G \times G$

ing proposition shows that $\mathscr{T}$-symmetries transport local classical dynamical system symmetries and hence uncovers the relationship of $\mathscr{T}$-symmetries and local symmetries. Note that the following proposition is connected to Lemma 3.2.11 and Corollary 3.2.12.

3.2.13 Proposition. *Let $\Psi_{\mathscr{T}}$ be a dynamical $\mathscr{T}$-system with $\mathscr{T}$-symmetry group $\mathfrak{S}$. Then for every $\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S}$ and $\lambda \in \Lambda$, S_λ and $S_{\pi^{-1}(\lambda)}$ are conjugated, i. e. there exists a subgroup $H_{\pi,\lambda} \leq G_\lambda$ such that $h_\lambda^{-1} S_\lambda h_\lambda = S_{\pi^{-1}(\lambda)}$ for all $h_\lambda \in H_{\pi,\lambda}$. Moreover, for a $\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S}$, one has*

$$g_\lambda \in S_\lambda \quad \textit{if and only if} \quad \pi \in C_{\mathcal{F},\lambda} \tag{3.54}$$

and

$$g \in \prod_{\lambda \in \Lambda} S_\lambda \quad \textit{if and only if} \quad \pi \in \bigcap_{\lambda \in \Lambda} C_{\mathcal{F},\lambda}. \tag{3.55}$$

In particular, (id_Λ, g) is a $\mathscr{T}$-symmetry of $\Psi_{\mathscr{T}}$ if and only if $g \in \mathrm{Fix}_{\widehat{H}}(\widehat{\mathfrak{S}}) \cap \prod_{\lambda \in \Lambda} S_\lambda$.

Proof. Let $g_\lambda \in S_\lambda$ be a symmetry of the dynamical system Ψ_λ, i. e., $g_\lambda \circ \mathcal{F}(\lambda) = \mathcal{F}(\lambda) \circ g_\lambda$, and $(\pi, h) \in \mathfrak{S}$ a $\mathscr{T}$-symmetry. For $k_\lambda = h_\lambda^{-1} g_\lambda h_\lambda$ we see that

$$\begin{aligned} k_\lambda \circ \mathcal{F}(\pi^{-1}(\lambda)) &= h_\lambda^{-1} \circ g_\lambda \circ \mathcal{F}(\lambda) \circ h_\lambda \\ &= h_\lambda^{-1} \circ \mathcal{F}(\lambda) \circ g_\lambda \circ h_\lambda \\ &= \mathcal{F}(\pi^{-1}(\lambda)) \circ h_\lambda^{-1} \circ g_\lambda \circ h_\lambda \\ &= \mathcal{F}(\pi^{-1}(\lambda)) \circ k_\lambda \end{aligned}$$

and, consequently, $k_\lambda \in S_{\pi^{-1}(\lambda)} \leq G_{\pi^{-1}(\lambda)}$, where we note once more that $G_{\pi^{-1}(\lambda)} = G_\lambda$. Thus,

$$h_\lambda^{-1} S_\lambda h_\lambda \leq S_{\pi^{-1}(\lambda)}$$

for $h_\lambda \in H_{\pi,\lambda} = \{h_\lambda \in G_\lambda |\ (\pi, h) \in \mathfrak{S}\}$.[2] Similarly, one has $h_\lambda S_{\pi^{-1}(\lambda)} h_\lambda^{-1} \leq S_\lambda$ implying $S_{\pi^{-1}(\lambda)} \leq h_\lambda^{-1} S_\lambda h_\lambda$ and finally $h_\lambda^{-1} S_\lambda h_\lambda = S_{\pi^{-1}(\lambda)}$.

Now consider a $\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S}$ with the special property that there is an index $\lambda \in \Lambda$ such that g_λ is a symmetry of the according dynamical system

[2]Observe that $H_{\pi,\lambda}$ is nothing but $(\widehat{H_\pi})_\lambda$ with $H_\pi = \widehat{\cdot}^{-1}(\pi) \cap \mathfrak{S}$. See also Lemma 3.2.11.

Ψ_λ, i. e. $g_\lambda \in S_\lambda$, which is equivalent to the equivariance relation

$$g_\lambda \circ \mathcal{F}(\lambda) = \mathcal{F}(\lambda) \circ g_\lambda.$$

By the $\mathscr{T}$-symmetry property, the weak equivariance equation

$$g_\lambda \circ \mathcal{F}(\pi^{-1}(\lambda)) = \mathcal{F}(\lambda) \circ g_\lambda$$

coexists. These two equations are easily seen to harmonize if and only if $\mathcal{F}(\pi^{-1}(\lambda)) = \mathcal{F}(\lambda)$ or $\pi \in C_{\mathcal{F},\lambda}$. Especially, if $g \in \prod_{\lambda\in\Lambda} S_\lambda$ meaning that each component g_λ of g is a symmetry of the corresponding system Ψ_λ, this is equivalent to $\pi \in C_{\mathcal{F},\lambda}$ for every $\lambda \in \Lambda$ or $\pi \in \bigcap_{\lambda\in\Lambda} C_{\mathcal{F},\lambda}$. Noting that $\mathrm{id}_\Lambda \in \bigcap_{\lambda\in\Lambda} C_{\mathcal{F},\lambda}$, we see that $(\mathrm{id}_\Lambda, g) \in \mathfrak{S}$ if and only if g additionally fulfills the compatibility condition $\pi^* g = g$ for all $\pi \in \widehat{\mathfrak{S}}$, i. e., $g \in \mathrm{Fix}_{\widehat{H}}(\widehat{\mathfrak{S}}) \cap \prod_{\lambda\in\Lambda} S_\lambda$. ■

We finish the coarse structural treatment of $\mathscr{T}$-symmetries by returning to Example 3.2.6 and reconsider it in the light of Proposition 3.2.13.

3.2.14 Example. We have seen in Example 3.2.6 that the vector fields

$$\mathcal{F}(1), \mathcal{F}(2) : \left(\mathbb{R}^4\right)^2 \to \left(\mathbb{R}^4\right)^2$$

of the dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ are of the following form:

$$\begin{aligned} \mathcal{F}(1)(y_1, y_2) &\overset{(3.26)}{=} \begin{pmatrix} g(y_1, y_2) \\ f(y_2, y_1) \end{pmatrix} \\ &\overset{(3.30),(3.34)}{=} \begin{pmatrix} B_1 & B_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h(y_1) \\ h(y_2) \end{pmatrix} \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} \mathcal{F}(2)(y_1, y_2) &\overset{(3.26)}{=} \begin{pmatrix} f(y_1, y_2) \\ g(y_2, y_1) \end{pmatrix} \\ &\overset{(3.30),(3.34)}{=} \begin{pmatrix} A_1 & A_2 \\ B_2 & B_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h(y_1) \\ h(y_2) \end{pmatrix}. \end{aligned} \tag{3.57}$$

With

$$C_1 = \begin{pmatrix} -(0.39 + \zeta) & -0.4 \\ 0.04 & -(0.39 + \zeta) \end{pmatrix}, \; C_2 = \begin{pmatrix} -(0.39 + 2\zeta) & -0.4 \\ 0.04 & -(0.39 + 2\zeta) \end{pmatrix}, \tag{3.58}$$

$\mathbb{I}_2$ denoting the identity matrix and 0 the zero matrix on $\mathbb{R}^2$, we can write the matrices $A_1, A_2, B_1, B_2 \in \mathbb{R}^{4\times 4}$ as block matrices

$$A_1 = \begin{pmatrix} C_2 & \zeta\mathbb{I}_2 \\ \zeta\mathbb{I}_2 & C_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \zeta\mathbb{I}_2 \\ 0 & 0 \end{pmatrix} \tag{3.59}$$

and

$$B_1 = \begin{pmatrix} C_1 & \zeta\mathbb{I}_2 \\ \zeta\mathbb{I}_2 & C_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ \zeta\mathbb{I}_2 & 0 \end{pmatrix}. \tag{3.60}$$

Furthermore, let $y_1 = (z_1, z_2)$, $y_2 = (z_3, z_4)$ and $z = (z_1, z_2, z_3, z_4) \in (\mathbb{R}^2)^4 \cong \mathbb{R}^8$ and let us define

$$H : (\mathbb{R}^2)^4 \to (\mathbb{R}^2)^4, \quad H(z_1, z_2, z_3, z_4) = \left(\widetilde{h}(z_1), \widetilde{h}(z_2), \widetilde{h}(z_3), \widetilde{h}(z_4)\right)^T \tag{3.61}$$

with

$$\widetilde{h}(x, y) = \begin{pmatrix} \epsilon x y^2 \\ 2.5\epsilon x y \end{pmatrix}. \tag{3.62}$$

Thereby, $\mathcal{F}(1)$ and $\mathcal{F}(2)$ can be considered as vector fields $(\mathbb{R}^2)^4 \to (\mathbb{R}^2)^4$ given by

$$\mathcal{F}(1)(y_1, y_2) = \mathcal{F}(1)(z) = \mathcal{A}z + H(z) \tag{3.63}$$

and

$$\mathcal{F}(2)(y_1, y_2) = \mathcal{F}(2)(z) = \mathcal{B}z + H(z) \tag{3.64}$$

with

$$\mathcal{A} = \begin{pmatrix} B_1 & B_2 \\ A_2 & A_1 \end{pmatrix} = \begin{pmatrix} C_1 & \zeta\mathbb{I}_2 & 0 & 0 \\ \zeta\mathbb{I}_2 & C_2 & \zeta\mathbb{I}_2 & 0 \\ 0 & \zeta\mathbb{I}_2 & C_2 & \zeta\mathbb{I}_2 \\ 0 & 0 & \zeta\mathbb{I}_2 & C_1 \end{pmatrix} \tag{3.65}$$

and

$$\mathcal{B} = \begin{pmatrix} A_1 & A_2 \\ B_2 & B_1 \end{pmatrix} = \begin{pmatrix} C_2 & \zeta\mathbb{I}_2 & 0 & \zeta\mathbb{I}_2 \\ \zeta\mathbb{I}_2 & C_1 & 0 & 0 \\ 0 & 0 & C_1 & \zeta\mathbb{I}_2 \\ \zeta\mathbb{I}_2 & 0 & \zeta\mathbb{I}_2 & C_2 \end{pmatrix}, \tag{3.66}$$

respectively. Viewing the vector fields $\mathcal{F}(1)$ and $\mathcal{F}(2)$ as maps $(\mathbb{R}^2)^4 \to (\mathbb{R}^2)^4$, the $\mathscr{T}$-symmetry (π, ι, ι) of $\Psi_{\mathscr{T}}$ appears as the element $\Upsilon_1 = (\pi, g_1, g_1)$ with π

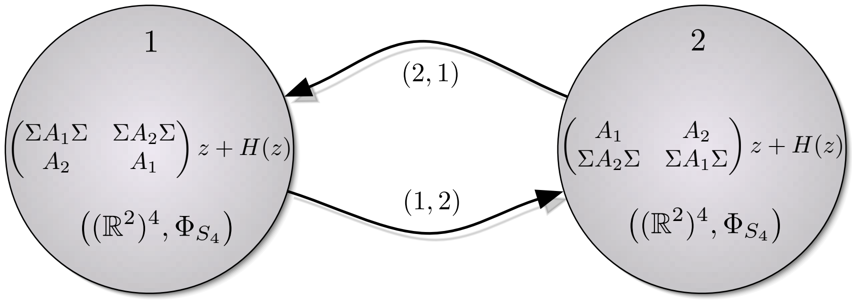


Figure 3.5: Dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ with vector fields (2.4) and (2.5) from Example 2.1.3 with $\mathscr{T}$-symmetries $\mathfrak{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

unchanged and $g_1 = (13)(24) \in S_4$. Furthermore, we find for $g_2 = (14)(23) \in S_4$

$$g_2\mathcal{A}z = \begin{pmatrix} \zeta z_3 + C_1 z_4 \\ \zeta z_2 + C_2 z_3 + \zeta z_4 \\ \zeta z_1 + C_2 z_2 + \zeta z_3 \\ C_1 z_1 + \zeta z_2 \end{pmatrix} = \begin{pmatrix} C_1 & \zeta\mathbb{I}_2 & 0 & 0 \\ \zeta\mathbb{I}_2 & C_2 & \zeta\mathbb{I}_2 & 0 \\ 0 & \zeta\mathbb{I}_2 & C_2 & \zeta\mathbb{I}_2 \\ 0 & 0 & \zeta\mathbb{I}_2 & C_1 \end{pmatrix} \begin{pmatrix} z_4 \\ z_3 \\ z_2 \\ z_1 \end{pmatrix} = \mathcal{A}g_2 z,$$

i. e. g_2 is a classical symmetry of $\mathcal{A}$ and thus of $\mathcal{F}(1)$ (since H is S_4-equivariant). Note that this symmetry corresponds to a reflection of the matrix entries of the block matrix $\mathcal{A}$ with respect to the anti-diagonal which in turn originates from the relation

$$\Sigma A_i = B_i \Sigma \quad i = 1, 2\,, \quad \text{with} \quad \Sigma = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}. \tag{3.67}$$

Now with

$$\widetilde{\Sigma} = \begin{pmatrix} 0 & \Sigma \\ \Sigma & 0 \end{pmatrix} \tag{3.68}$$

being the matrix representation of $g_2 \in S_4$ acting on $(\mathbb{R}^2)^4$, we have

$$\widetilde{\Sigma}\mathcal{A} = \begin{pmatrix} \Sigma A_2 & \Sigma A_1 \\ \Sigma B_1 & \Sigma B_2 \end{pmatrix} \overset{(3.67)}{=} \begin{pmatrix} B_2\Sigma & B_1\Sigma \\ A_1\Sigma & A_2\Sigma \end{pmatrix} = \mathcal{A}\widetilde{\Sigma}. \tag{3.69}$$

Hence, the dynamical system $\Psi_1 = ((\mathbb{R}^2)^4, \mathcal{F}(1))$ has the symmetry group $S_1 = \langle g_2 \rangle \cong \mathbb{Z}_2$. According to Proposition 3.2.13, $\mathscr{T}$-symmetries transport classical symmetries. In our case, this means that $g_2' = g_1^{-1} g_2 g_1$ is a classical symmetry of $\Psi_2 = ((\mathbb{R}^2)^4, \mathcal{F}(2))$. We obtain

$$g_2' = (13)(24)(14)(23)(13)(24) = (14)(23) = g_2, \tag{3.70}$$

since g_1 is self-inverse and commutes with g_2. Thus g_2 turns out to be a symmetry of Ψ_2, as well, and we get $S_2 = S_1 \cong \mathbb{Z}_2$. Due to Proposition 3.2.13, we also know that $\Upsilon_2 = (\mathrm{id}_\Lambda, g_2, g_2)$ is another $\mathscr{T}$-symmetry of $\Psi_{\mathscr{T}}$. For algebraic reasons or more precisely on the grounds of Proposition 3.2.7, which says that the $\mathscr{T}$-symmetries form a group, we know that

$$\Upsilon_3 = \Upsilon_1 \Upsilon_2 = (\pi, g_1, g_1)\,(\mathrm{id}_\Lambda, g_2, g_2) = (\pi, g_3, g_3) \tag{3.71}$$

with $g_3 = g_1 g_2 = (13)(24)(14)(23) = (12)(34)$ is another $\mathscr{T}$-symmetry of $\Psi_{\mathscr{T}}$. Thus, we end up with the $\mathscr{T}$-symmetry group

$$\begin{aligned}
\mathfrak{G} &= \langle \Upsilon_1, \Upsilon_2 \rangle \\
&= \{1, \Upsilon_1, \Upsilon_2, \Upsilon_3\} \\
&= \{(\mathrm{id}_\Lambda, \mathrm{id}, \mathrm{id})\,, (\pi, (13)(24), (13)(24))\,, (\mathrm{id}_\Lambda, (14)(23), (14)(23))\,, \\
&\quad\ (\pi, (12)(34), (12)(34))\} \\
&\cong \mathbb{Z}_2 \times \mathbb{Z}_2 .
\end{aligned} \tag{3.72}$$

In particular, all $\mathscr{T}$-symmetries are of order two and $\mathfrak{G}$ is commutative but not cyclic. ♢

As we will pass from dynamical $\mathscr{T}$-systems to hybrid dynamical systems by comprising the transitional dynamics directed by guards and resets in the next section, we will go ahead towards hybrid symmetries which heavily build on $\mathscr{T}$-symmetries and will provide dynamical meaning for the abstract objects we have dealt with above.

3.3 Hybrid Dynamical Systems and Hybrid Symmetries

It is the task of this section to make the rather abstract definitions and considerations of the preceding section amenable to the analysis of hybrid dynamical systems.

Having established a symmetry notion for extremely general and hence abstract dynamical system networks - the dynamical $\mathscr{T}$-systems -, which already possesses hybrid traits in the sense that graph and dynamical system symmetries are non-trivially interconnected, we are in the situation to take the remaining structure into account which distinguishes $\mathscr{T}$-systems and hybrid dynamical systems and thereby complete a truly hybrid symmetry framework. For this purpose, we discuss guards and resets in detail, as well as the algebraic structure of the emerging *hybrid symmetries*. As it will become evident in the course of the forthcoming investigations, $\mathscr{T}$-symmetries represent the structural core of hybrid symmetries underlining their importance once more.

Let us consider a hybrid dynamical system $\mathscr{H} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \mathcal{C}, \mathcal{G}, \mathcal{R})$. Equipping the phase spaces $\mathcal{D}(\lambda)$ with group actions Φ_{G_λ}, we can perceive the hybrid dynamical system $\mathscr{H}$ as a dynamical $\mathscr{T}$-system completed by clocks, guards and resets. Consequently, from the converse point of view, we can interpret $\mathscr{H}$ to be modeled on the $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ exhibiting the $\mathscr{T}$-symmetry group $\mathfrak{S}$. Having said this, from now on every hybrid dynamical system that we encounter is assumed to be accompanied by group actions Φ_{G_λ}, which is why we write $\mathscr{H} = (\Lambda, \mathcal{E}, \Theta, \mathcal{F}, \mathcal{C}, \mathcal{G}, \mathcal{R})$, henceforth.

We intend to establish a connection between $\mathfrak{S}$ and the instances $\mathcal{C}, \mathcal{G}, \mathcal{R}$, which determine the transitional dynamics of $\mathscr{H}$. Let $\mathcal{U} = \{U_e\}_{e \in \mathcal{E}}$ be a family of $\mathcal{E}$-indexed phase subspaces of the form $U_e \subset D_{\mathbb{T},\mathfrak{s}e} \times D_{\mathbb{T},\mathfrak{t}e}$. The action of G_λ on D_λ extends to an action on the extended phase space $D_{\mathbb{T},\lambda} = D_\lambda \times \mathbb{T}$ simply by

$$g_\lambda(x_\lambda, t_\lambda) = (g_\lambda x_\lambda, t_\lambda). \tag{3.73}$$

This means that - on the spatial component - G_λ acts as before while leaving the temporal component unaffected. In this sense, for a $\mathscr{T}$-symmetry $(\pi, g) \in \mathfrak{S} \leq \mathrm{Aut}(\mathscr{T}) \times \prod_{\lambda \in \Lambda} G_\lambda$ we define

$$(\pi, g)U_e := g_{\pi^{-1}(e)} U_{\pi^{-1}(e)}, \tag{3.74}$$

where $g_e = (g_{\mathfrak{s}e}, g_{\mathfrak{t}e}) \in G_{\mathfrak{s}e} \times G_{\mathfrak{t}e}$ acts component-wise on $D_{\mathbb{T},\mathfrak{s}e} \times D_{\mathbb{T},\mathfrak{t}e}$. In case $U_e \subset D_{\mathbb{T},\mathfrak{s}e}$, Equation (3.74) reduces to

$$(\pi, g)U_e := g_{\pi^{-1}(\mathfrak{s}e)} U_{\pi^{-1}(e)}. \tag{3.75}$$

It is now that we introduce the concept of hybrid symmetries by making use of Eqs. (3.74) and (3.75) in order to bring together $\mathscr{T}$-symmetries, guards and resets.

3.3.1 Definition (Hybrid Symmetries). Let $\mathscr{H} = (\Lambda, \mathcal{E}, \Theta, \mathcal{F}, \mathcal{T}, \mathcal{G}, \mathcal{R})$ be a hybrid dynamical system with underlying dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}} = (\mathscr{T}, \Theta, \mathcal{F})$. An element $\Upsilon = (\pi, g)$ of $H = \mathrm{Aut}(\mathscr{T}) \times \prod_{\lambda\in\Lambda} G_\lambda$ is a *hybrid symmetry* of $\mathscr{H}$ if it is a $\mathscr{T}$-symmetry of $\Psi_{\mathscr{T}}$ and leaves the guards $\mathcal{G}(e)$ and resets $\mathcal{R}(e)$ invariant:

$$\Upsilon\mathcal{G}(e) = \mathcal{G}(e) \qquad \text{and} \qquad \Upsilon\mathcal{R}(e) = \mathcal{R}(e) \tag{3.76}$$

for all discrete transitions $e \in \mathcal{E}$. ♢

Before analyzing the algebraic structure of hybrid symmetries, we take a closer look on the invariance condition imposed on the resets. While the guards are simply subsets of the extended phase spaces $D_{\mathbb{T},\lambda}$, the resets $\mathcal{R}(e) \subset D_{\mathbb{T},\mathfrak{s}e} \times D_{\mathbb{T},\mathfrak{t}e}$ as introduced in Definition 2.1.1 may be interpreted as possibly set-valued maps (as already hinted at in Remark 2.1.2). Keeping this in mind, we trace the meaning of reset invariance and see that it translates to a weak form of equivariance when considering the resets as maps rather than as graphs of maps. Viewed as (generally set-valued) maps, the resets $\mathcal{R}(e) \subset G_e \times D_{\mathbb{T},\mathfrak{t}e}$ take the form

$$R_e : G_e \multimap D_{\mathbb{T},\mathfrak{t}e}, \quad (\mathfrak{s}e, x, t) \mapsto \left(\mathfrak{t}e, \bar{R}_e(x), 0\right), \tag{3.77}$$

where $\bar{R}_e : D_{\mathfrak{s}e} \to D_{\mathfrak{t}e}$ is the essential reset of the spatial component.

3.3.2 Lemma. *For a $\mathscr{T}$-symmetry $\Upsilon = (\pi, g) \in \mathfrak{S}$, the invariance with respect to Υ, i. e. $\Upsilon\mathcal{R}(e) = \mathcal{R}(e)$ for all $e \in \mathcal{E}$, is equivalent to the weak equivariance*

$$\bar{R}_e \circ g_{\mathfrak{s}e} = g_{\mathfrak{t}e} \circ \bar{R}_{\pi^{-1}(e)}. \tag{3.78}$$

Proof. The invariance condition $\Upsilon\mathcal{R}(e) = g_{\pi^{-1}(e)}\mathcal{R}(\pi^{-1}(e)) = \mathcal{R}(e)$ tells us that for each $y \in \mathcal{R}(\pi^{-1}(e))$ the point $g_{\pi^{-1}(e)}y$ represents an element in $\mathcal{R}(e)$. Explicitely, we have

$$\begin{aligned} & g_{\pi^{-1}(e)}\left((\pi^{-1}(\mathfrak{s}e), x, t), (\pi^{-1}(\mathfrak{t}e), \bar{R}_{\pi^{-1}(e)}(x), 0)\right) \\ = \; & \left((\pi^{-1}(\mathfrak{s}e), g_{\pi^{-1}(\mathfrak{s}e)}x, t), (\pi^{-1}(\mathfrak{t}e), g_{\pi^{-1}(\mathfrak{t}e)}\bar{R}_{\pi^{-1}(e)}(x), 0)\right) \\ \in \; & G_{\pi^{-1}(e)} \times D_{\mathbb{T},\pi^{-1}(\mathfrak{t}e)}. \end{aligned}$$

Now, in order to perceive this point as an element of $\mathcal{R}(e)$, using $\pi^* g = g$, we have to conclude that $g_{\mathfrak{t}e} \circ \bar{R}_{\pi^{-1}(e)} = \bar{R}_e \circ g_{\mathfrak{s}e}$ giving us the weak equivariance as stated above. ■

We now turn to the algebraic structure of hybrid symmetries: While Proposition 3.2.7 tells us that the $\mathscr{T}$-symmetries $\mathfrak{S}$ form a group, it is not obvious at first sight that the additional invariance of guards and resets does indeed preserve the algebraic structure of $\mathfrak{S}$. In fact, we find that it does when subjecting the matter to a formal treatment.

For a family $\mathcal{U} = \{U_e\}_{e\in\mathcal{E}}$, as above, we define the *$\mathcal{U}$-stabilizer* $\Sigma_{\mathcal{U}}$ to be the set

$$\Sigma_{\mathcal{U}} = \{\Upsilon \in \mathfrak{S} \mid \Upsilon U_e = U_e \quad \text{for all } e \in \mathcal{E}\}\,. \tag{3.79}$$

In the following, we uncover the algebraic structure of the stabilizer $\Sigma_{\mathcal{U}}$.

3.3.3 Lemma. *For a family $\mathcal{U} = \{U_e\}_{e\in\mathcal{E}}$, $\Sigma_{\mathcal{U}}$ is a subgroup of $\mathfrak{S}$ which acts on $\mathcal{U}$ in accordance with* (3.74) *and* (3.75).

Proof. Certainly, the neutral element $1_{\mathfrak{S}} = (\mathrm{id}_\Lambda, 1_{\widehat{H}})$ of $\mathfrak{S}$ is in $\Sigma_{\mathcal{U}}$. Let $\Upsilon_1 = (\pi_1, g_1)$ and $\Upsilon_2 = (\pi_2, g_2)$ be two elements of $\Sigma_{\mathcal{U}}$. Then, in particular, we have

$$\Upsilon_1 U_e = U_e \quad \text{and} \quad \Upsilon_2 U_{\pi_1^{-1}(e)} = U_{\pi_1^{-1}(e)},$$

or, equivalently (keeping in mind (3.74) and (3.75)),

$$U_e = (g_1)_{\pi_1^{-1}(e)} U_{\pi_1^{-1}(e)} \quad \text{and} \quad U_{\pi_1^{-1}(e)} = (g_2)_{\pi_2^{-1}(e)} U_{\pi_2^{-1}(\pi_1^{-1}(e))}.$$

Combining these and using $g_1, g_2 \in \mathrm{Fix}_{\widehat{\mathfrak{S}}}(\widehat{\mathfrak{S}})$, we end up with

$$U_e = (g_1)_{\pi_1^{-1}(e)} (g_2)_{\pi_2^{-1}(e)} U_{\pi_2^{-1}(\pi_1^{-1}(e))} = (g_1 g_2)_{(\pi_1\pi_2)^{-1}(e)}\, U_{(\pi_1\pi_2)^{-1}(e)} = (\Upsilon_1\Upsilon_2) U_e,$$

which shows that $\Upsilon_1\Upsilon_2 \in \Sigma_{\mathcal{U}}$. Let $\Upsilon = (\pi, g) \in \Sigma_{\mathcal{U}}$ and $e \in \mathcal{E}$. We set $e' = \pi(e)$. Since $\Upsilon U_{e'} = U_{e'}$, we note

$$U_{e'} = g_e U_e \tag{3.83}$$

and compute

$$\Upsilon^{-1} U_e = g_{\pi(e)}^{-1} U_{\pi(e)} \overset{e'=\pi(e)}{=} g_{e'}^{-1} U_{e'} \overset{(3.83)}{=} g_{e'}^{-1} g_e U_e.$$

Now, the constancy condition $\pi^* g = g$ implies that $g_e = g_{\pi^{-1}(e')} = g_{e'}$, which finally shows that $\Upsilon^{-1} \in \Sigma_{\mathcal{U}}$ and hence that $\Sigma_{\mathcal{U}}$ is indeed a subgroup of $\mathfrak{S}$. Furthermore, we observe that $\Upsilon_2(\Upsilon_1 U_e) = \Upsilon_2 U_e = U_e$ holds for all $e \in \mathcal{E}$, which yields $\Upsilon_2(\Upsilon_1 U_e) = (\Upsilon_2\Upsilon_1)U_e$ for all $\Upsilon_1, \Upsilon_2 \in \Sigma_{\mathcal{U}}$ and all $e \in \mathcal{E}$. Therefore, $\mathcal{U}$ is a $\Sigma_{\mathcal{U}}$-set. ■

Hence, the guard and reset stabilizer $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{R}}$ turn out to be subgroups of the $\mathscr{T}$-symmetry group $\mathfrak{S}$. Recall that in Remark 2.1.2, we pointed out that the guards are contained in the resets in a very natural manner. Thus, the question arises how the stabilizers $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{R}}$ are related to one another. We will answer this in the following.

3.3.4 Lemma. *The reset stabilizer $\Sigma_{\mathcal{R}}$ is a subgroup of the guard stabilizer $\Sigma_{\mathcal{G}}$.*

Proof. Bearing in mind Remark 2.1.2, for $e \in \mathcal{E}$, we can write

$$R_e \subset (R_e)_{\mathfrak{s}e} \times (R_e)_{\mathfrak{t}e} \overset{(2.3)}{=} G_e \times (R_e)_{\mathfrak{t}e}$$

and, accordingly, for an automorphism $\pi \in \mathrm{Aut}(\mathscr{T})$, we find

$$R_{\pi^{-1}(e)} \subset G_{\pi^{-1}(e)} \times \left(R_{\pi^{-1}(e)}\right)_{\mathfrak{t}\pi^{-1}(e)}.$$

Then, for an element $\Upsilon = (\pi, g) \in \Sigma_{\mathcal{R}}$, we have

$$R_e = \Upsilon R_e = g_e R_{\pi^{-1}(e)} \subset \left(g_{\mathfrak{s}e} G_{\pi^{-1}(e)}\right) \times \left(g_{\mathfrak{t}e}\left(R_{\pi^{-1}(e)}\right)_{\mathfrak{t}\pi^{-1}(e)}\right), \tag{3.87}$$

which – via projection – implies the guard invariance

$$G_e \overset{(2.3)}{=} (R_e)_{\mathfrak{s}e} \overset{(3.87)}{=} \left(g_e R_{\pi^{-1}(e)}\right)_{\mathfrak{s}e} \overset{(2.3)}{=} g_{\mathfrak{s}e} G_{\pi^{-1}(e)} = \Upsilon G_e.$$

Hence, $\Upsilon \in \Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{R}}$ turns out to be a subgroup of $\Sigma_{\mathcal{G}}$. ■

3.3.5 Corollary. *For $e \in \mathcal{E}$ and $\Upsilon = (\pi, g) \in \Sigma_{\mathcal{R}}$, one has*

$$g_{\mathfrak{t}e}^{-1}(R_e)_{\mathfrak{t}e} = \left(R_{\pi^{-1}(e)}\right)_{\pi^{-1}(\mathfrak{t}e)}. \tag{3.89}$$

Proof. Consider Equation (3.87) and observe

$$(R_e)_{\mathfrak{t}e} = g_{\mathfrak{t}e}\left(R_{\pi^{-1}(e)}\right)_{\mathfrak{t}\pi^{-1}(e)}.$$

Using the commutativity $\mathfrak{t} \circ \pi^{-1} = \pi^{-1} \circ \mathfrak{t}$ we end up with the identity stated above. ■

The following lemma provides us with a class of hybrid systems for which the stabilizers $\Sigma_{\mathcal{G}}$ and $\Sigma_{\mathcal{R}}$ coincide.

3.3.6 Lemma. *Let $\mathscr{H}$ be a hybrid dynamical system with $\mathscr{T}$-symmetries $\mathfrak{S}$. If the resets $\mathcal{R}(e)$ are trivial (meaning $\mathcal{R}(e) = \mathrm{id}_{\mathcal{G}(e)}$ for all $e \in \mathcal{E}$), and for all discrete transitions $e \in \mathcal{E}$ the source $\mathfrak{s}e$ and the target $\mathfrak{t}e$ are similar as vertices of the transition graph $\mathscr{T}$, then $\Sigma_{\mathcal{R}} = \Sigma_{\mathcal{G}}$.*

Proof. Since Lemma 3.3.4 states the subgroup relation $\Sigma_{\mathcal{R}} \leq \Sigma_{\mathcal{G}}$, it suffices to show the converse $\Sigma_{\mathcal{G}} \leq \Sigma_{\mathcal{R}}$. Reviewing the proof of Lemma 3.3.4 and using Corollary 3.3.5, we observe that it remains to verify the identity

$$\left(\mathcal{R}(e)\right)_{\mathfrak{t}e} = g_{\mathfrak{t}e} \left(\mathcal{R}(\pi^{-1}(e))\right)_{\pi^{-1}(\mathfrak{t}e)} \tag{3.90}$$

for each $(\pi, g) \in \Sigma_{\mathcal{G}}$. Now, let (π, g) be an element of $\Sigma_{\mathcal{G}}$. By definition, we have

$$\mathcal{G}(e) = g_{\mathfrak{s}e}\mathcal{G}(\pi^{-1}(e)). \tag{3.91}$$

The triviality of the resets enforces

$$(\mathcal{R}(e))_{\mathfrak{t}e} = \mathcal{G}(e) \tag{3.92}$$

for all $e \in \mathcal{E}$, especially for $\pi^{-1}(e)$. The similarity of source $\mathfrak{s}e$ and target $\mathfrak{t}e$ encodes the fact that $\mathfrak{s}e$ and $\mathfrak{t}e$ belong to the same $\widehat{\mathfrak{S}}$-orbit and, thus, induces the equality

$$g_{\mathfrak{s}e} = g_{\mathfrak{t}e}. \tag{3.93}$$

Now, by means of (3.92) and (3.93), Equation (3.91) translates to the desired identity (3.90). Ultimately, we find $(\pi, g) \in \Sigma_{\mathcal{R}}$ and, consequently, $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{R}}$. ■

As a matter of fact, the additional guard and reset invariance preserves the group structure provided by the $\mathscr{T}$-symmetry group $\mathfrak{S}$ and thus completes the construction of a global symmetry organizing object for hybrid dynamical systems. Furthermore, it should be pointed out that so far the group $\mathrm{Aut}(\mathscr{T}) \times \prod_{\lambda \in \Lambda} G_\lambda$ is understood to simultaneously act on all phase spaces involved. We will see that the action of the hybrid symmetry group $\mathcal{H}$ lifts to an action on the hybrid phase space $D \subset \Lambda \times \mathbb{R}^n$, which opens up the possibility to analyze hybrid dynamics in the presence of hybrid symmetries.

3.3.7 Theorem. *The hybrid symmetries of a hybrid dynamical system* $\mathscr{H}$ *form a group, the hybrid symmetry group* $\mathcal{H}$*, which acts on the hybrid phase space* $D = \bigcup_{\lambda\in\Lambda} \mathcal{D}(\lambda) \subset \Lambda \times \mathbb{R}^n$ *via the action*

$$\mathcal{H} \times D \to D, \quad ((\pi, g), (\lambda, x)) \mapsto \left(\pi^{-1}(\lambda), g^{-1}_{\pi^{-1}(\lambda)} x\right). \tag{3.94}$$

Proof. By means of Proposition 3.2.7 and Lemma 3.3.3, we are aware of the fact that $\mathfrak{S}$, the guard stabilizer $\Sigma_{\mathcal{G}}$ and the reset stabilizer $\Sigma_{\mathcal{R}}$ are subgroups of $H = \mathrm{Aut}\,(\mathscr{T}) \times \prod_{\lambda\in\Lambda} G_\lambda$. By Definition 3.3.1, the collection $\mathcal{H}$ of all hybrid symmetries is given by the intersection

$$\mathcal{H} = \mathfrak{S} \cap \Sigma_{\mathcal{G}} \cap \Sigma_{\mathcal{R}} \tag{3.95}$$

of groups which itself features the structure of a group. Note that due to Lemma 3.3.4, we have $\mathcal{H} = \Sigma_{\mathcal{R}}$, i. e. the reset stabilizer encodes the complete hybrid symmetry structure of the hybrid dynamical system. In order to provide evidence that the group $\mathcal{H}$ acts on $D = \bigcup_{\lambda\in\Lambda} \mathcal{D}(\lambda) \subset \Lambda \times \mathbb{R}^n$ via the hybrid action specified in (3.94), we first of all note that $1_{\mathcal{H}}(\lambda, x) = (\lambda, x)$ for all $(\lambda, x) \in D$ where $1_{\mathcal{H}} = (\mathrm{id}_\Lambda, 1_{\widehat{H}})$ is the neutral element of $\mathcal{H}$. Secondly, for two hybrid symmetries $\Upsilon_1 = (\pi_1, g_1), \Upsilon_2 = (\pi_2, g_2) \in \mathcal{H}$, we compute

$$\begin{aligned}
\Upsilon_2\left(\Upsilon_1(\lambda, x)\right) &\overset{(3.94)}{=} \left(\pi_1^{-1}\left(\pi_2^{-1}(\lambda)\right), (g_1)^{-1}_{\pi_1^{-1}(\lambda)}\,(g_2)^{-1}_{\pi_1^{-1}\left(\pi_2^{-1}(\lambda)\right)}\,x\right) \\
&\overset{\pi^* g\, =\, g}{=} \left(\left(\pi_2\pi_1\right)^{-1}(\lambda), \left(g_2 g_1\right)^{-1}_{(\pi_2\pi_1)^{-1}(\lambda)}\,x\right) \\
&\overset{(3.94)}{=} \left(\pi_2\pi_1, g_2 g_1\right)(\lambda, x) \\
&= (\Upsilon_2\Upsilon_1)(\lambda, x),
\end{aligned}$$

which documents the $\mathcal{H}$-action on D and completes the proof. ■

3.3.8 Remark. The proof of Theorem 3.3.7 shows that the hybrid symmetries $\mathcal{H}$ form a subgroup of the $\mathscr{T}$-symmetries $\mathfrak{S}$. For this reason, all structural statements of Section 3.2 immediately carry over to hybrid symmetries and thus can be directly applied to general hybrid dynamical systems which exhibit symmetries. ♢

In order to conclude this section, we reconsider Example 2.1.3 from the viewpoint of hybrid symmetries.

3.3.9 Example. We know from Example 3.2.6 that the underlying dynamical $\mathscr{T}$-system $\Psi_{\mathscr{T}}$ has as special fine structure in terms of the maps f and g (cf. (3.30) and (3.34)) which is also indicated in Fig. 3.6. Moreover, $\Psi_{\mathscr{T}}$ possesses $\mathfrak{S} = \langle(\pi, \iota, \iota)\rangle \cong \mathbb{Z}_2$ as its $\mathscr{T}$-symmetry group with transition graph automorphism $\pi = (12)$ interchanging the discrete states 1 and 2 and the involution $\iota : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4 \times \mathbb{R}^4$ that interchanges arguments. We now focus on guards and resets. Recall that the guards are given by $\mathcal{G}(1,2) = \mathcal{G}(2,1) = \mathbb{R}^8 \times \{20\}$. For $\Upsilon = (\pi, \iota, \iota)$, we find that

$$\Upsilon\mathcal{G}(1,2) = \iota\mathcal{G}(\pi^{-1}(1), \pi^{-1}(2)) = \iota\mathcal{G}(2,1) = \iota(\mathbb{R}^8 \times \{20\}) = \mathbb{R}^8 \times \{20\} = \mathcal{G}(1,2)$$

since ι acts trivially on the temporal component by (3.73). Thus, $\Upsilon\mathcal{G}(e) = \mathcal{G}(e)$ holds for all $e \in \mathcal{E}$ which implies $\Sigma_{\mathcal{G}} = \mathfrak{S}$. The resets are essentially given by identities, i. e. they are of the form $\mathcal{R}(e) = \{((\mathfrak{s}e, x, 20), (\mathfrak{t}e, x, 0)) \mid x \in \mathbb{R}^8\}$. Therefore, $\Upsilon\mathcal{R}(e) = \mathcal{R}(e)$ for all $e \in \mathcal{E}$ yielding $\Sigma_{\mathcal{R}} = \Sigma_{\mathcal{G}}$ and, thus, we obtain

$$\mathcal{H} = \mathfrak{S} \cong \mathbb{Z}_2 \tag{3.96}$$

for the hybrid symmetry group $\mathcal{H}$ of $\mathscr{H}$.

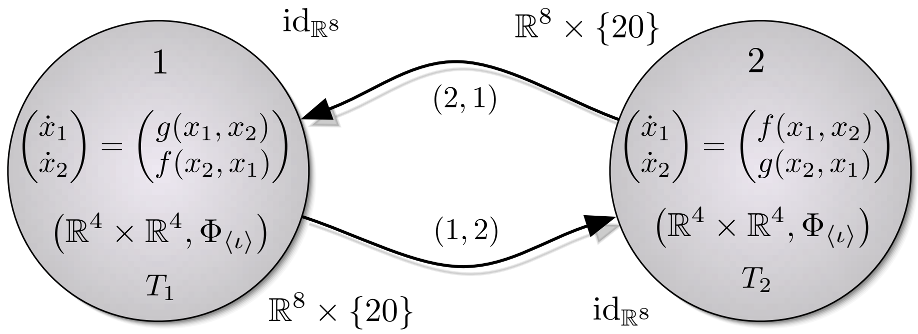


Figure 3.6: Hybrid dynamical system $\mathscr{H}$ with vector fields (2.4) and (2.5) from Example 2.1.3 with hybrid symmetries; cp. Fig. 2.2 and Fig. 3.2.

Note that this is a special instance of Lemma 3.3.6. ◇

3.4 Consequences of Hybrid Symmetries

After having developed a notion of symmetry for hybrid dynamical systems and having revealed its algebraic properties in the foregoing section, we intend to shed light on the immediate consequences of hybrid symmetries on hybrid dynamics. First of all, we focus on the implications for executions and show that hybrid symmetries preserve their characteristic structure specified by Definition 2.2.2. For this purpose, we prescribe the action of hybrid symmetries on hybrid trajectories as follows:

3.4.1 Definition. For an execution $\chi = (\tau, \gamma, x)$ of $\mathscr{H}$ and a hybrid symmetry $\Upsilon = (\pi, g) \in \mathcal{H}$ we define

$$\Upsilon\chi = (\tau, \pi\gamma, gx)\,, \tag{3.97}$$

with

$$\pi\gamma : [\tau] \to \Lambda, \quad (\pi\gamma)(k) = \pi^{-1}\left(\gamma(k)\right)$$

and

$$gx = \left\{g^{-1}_{\pi^{-1}(\gamma(k))} x_k : I_k \to \mathcal{D}(\pi^{-1}(\gamma(k)))\right\}_{k\in[\tau]}\,. \qquad \diamondsuit$$

Perceiving an execution χ of $\mathscr{H}$ as a map $\mathcal{T} \to D$ (cp. Eqs. (2.13) and (2.14)), for a hybrid symmetry $\Upsilon = (\pi, g) \in \mathcal{H}$, we obtain

$$\Upsilon\chi(k,t) = \left(\pi^{-1}(\gamma(k)), g^{-1}_{\pi^{-1}(\gamma(k))} x_k(t)\right) \tag{3.100}$$

for all $(k, t) \in T$, in accordance with the hybrid action of $\mathcal{H}$ on D (cf. Theorem 3.3.7). In the next step, we prove that - analogous to the classical case - hybrid symmetries transform executions to executions, i. e. they leave the property of being an execution invariant. Note that this is the first and at the same time most important evidence that the concept of hybrid symmetries as it has been set up in Definition 3.3.1 is adequate and not completely unfeasible.

3.4.2 Proposition ($\mathcal{H}$-Invariance of E). *Let $\mathscr{H}$ be a hybrid dynamical system with hybrid symmetry group $\mathcal{H}$. For every execution χ of $\mathscr{H}$ and for every hybrid symmetry $\Upsilon \in \mathcal{H}$, $\Upsilon\chi$ is an $\mathscr{H}$-execution, as well. In other words, the set E of all $\mathscr{H}$-executions is $\mathcal{H}$-invariant.*

Proof. Let $\chi : T \to D$ be an execution of $\mathscr{H}$ and $\Upsilon = (\pi, g) \in \mathcal{H}$ a hybrid symmetry. According to Definition 2.2.2, $x_k : I_k \to \mathcal{D}(\gamma(k))$ is a trajectory of the dynamical system $\Psi_{\gamma(k)} = (\mathcal{D}(\gamma(k)), \mathcal{F}(\gamma(k)))$, i.e. we have

$$\frac{\mathrm{d}}{\mathrm{dt}} x_k(t) = \mathcal{F}(\gamma(k))\,(x_k(t))\,. \tag{3.101}$$

For every $k \in [\tau]$, Lemma 3.2.5 provides the identity

$$g^{-1}_{\gamma(k)} \circ \mathcal{F}(\gamma(k)) = \mathcal{F}\left(\pi^{-1}(\gamma(k))\right) \circ g^{-1}_{\pi^{-1}(\gamma(k))}. \tag{3.102}$$

For the path $g^{-1}_{\pi^{-1}(\gamma(k))} x_k : I_k \to \mathcal{D}(\pi^{-1}(\gamma(k)))$, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left(g^{-1}_{\pi^{-1}(\gamma(k))} x_k(t)\right) &\overset{\pi^* g = g}{=} g^{-1}_{\gamma(k)}\left(\frac{\mathrm{d}}{\mathrm{dt}} x_k(t)\right) \\
&\overset{(3.101)}{=} g^{-1}_{\gamma(k)}\, \mathcal{F}(\gamma(k))\,(x_k(t)) \\
&\overset{(3.102)}{=} \mathcal{F}\left(\pi^{-1}(\gamma(k))\right)\left(g^{-1}_{\pi^{-1}(\gamma(k))} x_k(t)\right),
\end{aligned}$$

showing that $g^{-1}_{\pi^{-1}(\gamma(k))} x_k$ is indeed a trajectory of the dynamical system $\Psi_{\pi^{-1}(\gamma(k))} = \left(\Theta\left(\pi^{-1}(\gamma(k))\right), \mathcal{F}\left(\pi^{-1}(\gamma(k))\right)\right)$. Since $e_k := (\gamma(k), \gamma(k+1)) \in \mathcal{E}$ by assumption, application of the $\mathscr{T}$-automorphism π yields

$$\pi^{-1}(e_k) = \left(\pi^{-1}(\gamma(k)), \pi^{-1}(\gamma(k+1))\right) \in \mathcal{E}.$$

Moreover, $(x_k(\tau_k'), |I_k|) \in \mathcal{G}(e_k)$ implies

$$\left(g^{-1}_{\gamma(k)} x_k(\tau_k'), |I_k|\right) \in g^{-1}_{\gamma(k)} \mathcal{G}(e_k) = \mathcal{G}(\pi^{-1}(e_k)),$$

due to the fact that $\Upsilon \mathcal{G}(e_k) = \mathcal{G}(e_k)$ holds. Likewise, the reset relation

$$((x_k(\tau_k'), |I_k|), (x_{k+1}(\tau_{k+1}), 0)) \in \mathcal{R}(e_k)$$

in conjunction with the reset invariance $\Upsilon \mathcal{R}(e_k) = \mathcal{R}(e_k)$ ensures

$$\begin{aligned}
&\left((g^{-1}_{\gamma(k)} x_k(\tau_k'), |I_k|), (g^{-1}_{\gamma(k+1)} x_{k+1}(\tau_{k+1}), 0)\right) \\
\in\ &(g_{\gamma(k)}, g_{\gamma(k+1)})^{-1} \mathcal{R}(e_k) = \mathcal{R}(\pi^{-1}(e_k)),
\end{aligned}$$

ultimately verifying that $\Upsilon\chi$ is indeed an execution of $\mathscr{H}$. ∎

Note that – on the basis of Definition 3.4.1 – the hybrid time trajectory τ_χ of an execution $\chi = (\tau, \gamma, x) \in E$ is not affected by hybrid symmetries, i.e. $\tau_\chi = \tau_{\Upsilon\chi}$ for every hybrid symmetry $\Upsilon \in \mathcal{H}$. Since, contrariwise, the classification of executions covered by Definition 2.2.3 is solely based on the discrete and continuous length of hybrid time trajectories, hybrid symmetries have no effect on this partition.

Among others, the preceding result yields the fact that the reachability domain which is of utmost importance whenever a hybrid system's safety properties are analyzed behaves invariantly under hybrid symmetry transformations. This offers the possibility of reducing computational effort when computing $\mathrm{Reach}(\mathscr{H})$ by effective exploitation of known symmetry features.

3.4.3 Corollary ($\boldsymbol{\mathcal{H}}$-Invariance of Reach($\mathscr{H}$)). *Given an $\mathcal{H}$-symmetric hybrid dynamical system $\mathscr{H}$, its reachability domain* $\mathrm{Reach}(\mathscr{H}) \subset D$ *is $\mathcal{H}$-invariant.*

Proof. At first, by Theorem 3.3.7 the global hybrid phase space $D \subset \Lambda \times \mathbb{R}^n$ is an $\mathcal{H}$-space. By Definition 2.2.5, for every reachable state $(\lambda, x) \in \mathrm{Reach}\,(\mathscr{H})$, there exists a finite execution $\chi \in E^{<\infty}$ with final state $\odot(\chi) = (\lambda, x)$. Now, Proposition 3.4.2 asserts that $\Upsilon\chi \in E^{<\infty}$ for all $\Upsilon \in \mathcal{H}$. Observing $\odot(\Upsilon\chi) = \Upsilon \odot (\chi)$, we see that $\Upsilon(\lambda, x) \in \mathrm{Reach}\,(\mathscr{H})$. Thus, $\mathrm{Reach}\,(\mathscr{H})$ is $\mathcal{H}$-invariant. ∎

3.5 Hybrid Fixed-Point Spaces

Having seen that hybrid symmetries structurally preserve the dynamics of a hybrid dynamical system, we turn towards symmetry-induced subsets of the hybrid phase space, so-called *fixed-point spaces*. It is one of the pivotal points in the theory of equivariant dynamical systems that the fixed-point spaces induced by a subgroup of the symmetry group are flow-invariant (see [GSS88] and [GS02], for instance). In contrast to the classical case of equivariant vector fields we are involved with families of weakly equivariant vector fields. We cannot expect that hybrid objects behave as advantageous as in the smooth case; nonetheless we can ask for the structure and invariance properties of hybrid fixed-point spaces.

3.5.1 Definition (Hybrid Fixed-Point Space). Let $\mathscr{H}$ be a hybrid dynamical system with hybrid symmetry group $\mathcal{H}$. For a subgroup $\Xi \leq \mathcal{H}$ define

$$\mathrm{Fix}(\Xi) = \{p = (\lambda, x) \in D \mid \Upsilon p = p \text{ for all } \Upsilon \in \Xi\} \tag{3.106}$$

with respect to the action (3.94) of $\mathcal{H}$ on D. It is referred to as the *hybrid fixed-point space* of Ξ. ♢

Given a hybrid dynamical system $\mathscr{H}$, for each phase space $\mathcal{D}(\lambda)$ with according group action of G_λ and a subgroup $H_\lambda \leq G_\lambda$, we can consider the λ-*localized fixed-point space*

$$\mathrm{Fix}_\lambda(H_\lambda) = \{x \in D_\lambda \mid g_\lambda x = x \text{ for all } g_\lambda \in H_\lambda\} \subset D_\lambda, \tag{3.107}$$

which corresponds to the fixed-point space in the classical framework of dynamical systems. On the discrete side, given the transition graph $\mathscr{T} = (\Lambda, \mathcal{E})$ and a subgroup $\Gamma \leq \mathrm{Aut}(\mathscr{T})$, we can define the $\mathscr{T}$-*fixed-point set*

$$\mathrm{Fix}_{\mathscr{T}}(\Gamma) = \left\{\lambda \in \Lambda \,\middle|\, \pi^{-1}(\lambda) = \lambda \text{ for all } \pi \in \Gamma\right\} \subset \Lambda. \tag{3.108}$$

Making use of these concepts, we can describe hybrid fixed-point spaces in the following way.

3.5.2 Lemma. *For a subgroup $\Xi \leq \mathcal{H}$, one has*

$$\mathrm{Fix}(\Xi) = \bigcup_{\lambda \in \mathrm{Fix}_{\mathscr{T}}(\widehat{\Xi})} \{\lambda\} \times \mathrm{Fix}_\lambda(\widehat{\Xi}_\lambda), \tag{3.109}$$

where $\widehat{\Xi} \leq \mathrm{Aut}\,(\mathscr{T})$ and $\widehat{\Xi}_\lambda \leq G_\lambda$ denote the corresponding projections of Ξ.

Proof. Keeping in mind (3.94), for $p = (\lambda, x) \in \mathrm{Fix}(\Xi)$ and $\Upsilon = (\pi, g) \in \Xi$, we find

$$(\lambda, x) = p = \Upsilon p = \left(\pi^{-1}(\lambda), g^{-1}_{\pi^{-1}(\lambda)} x\right).$$

By means of the constancy condition $g \in \mathrm{Fix}_{\widehat{H}}(\widehat{\mathcal{H}})$, this implies $\pi^{-1}(\lambda) = \lambda$ for all $\pi \in \widehat{\Xi}$ as well as $g_\lambda x = x$ for all $g_\lambda \in \widehat{\Xi}_\lambda$. Thus, $\lambda \in \mathrm{Fix}_{\mathscr{T}}(\widehat{\Xi})$ and $x \in \mathrm{Fix}_\lambda(\widehat{\Xi}_\lambda)$ leading to the desired statement. ■

This lemma shows that hybrid fixed-point spaces are locally given by classical fixed-point spaces which are connected on the grounds of a subgraph of the transition graph $\mathscr{T}$ which is still to be specified. An execution staying in a fixed-point space forever is easily characterized by a straightforward application of Lemma 3.5.2.

3.5.3 Corollary. *Let $\mathscr{H}$ be a hybrid dynamical system with hybrid symmetries $\mathcal{H}$. Furthermore, let $\Xi \leq \mathcal{H}$ be a subgroup and χ an execution of $\mathscr{H}$. Then χ stays in* $\operatorname{Fix}(\Xi)$ *all the time if and only if*

$$\gamma(k) \in \operatorname{Fix}_{\mathscr{T}}(\widehat{\Xi}) \quad \textit{and} \quad x_k(t) \in \operatorname{Fix}_{\gamma(k)}(\widehat{\Xi}_{\gamma(k)}) \tag{3.110}$$

for all $(k,t) \in \mathcal{T}$.

The following considerations give attention to the surprising fact that hybrid fixed point spaces are not as invariant as they are celebrated to be in the classical dynamical system setting. In order to tackle the invariance properties of hybrid fixed-point spaces, we provide a simple characterization of hybrid invariance according to Definition 2.2.6 and afterwards introduce the concepts of *invariant subgraphs* and *output sets.*

We intend to characterize hybrid invariance in terms of the data a hybrid dynamical system is made of. Consider an invariant subset $\mathcal{S} \subset D$ and an execution χ of $\mathscr{H}$ starting in $\mathcal{S}$. Then for each $\lambda \in \Lambda$ and $k \in \gamma^{-1}(\lambda)$, one has $\chi(k,t) \in \mathcal{S} \cap \mathcal{D}(\lambda)$, otherwise $\mathcal{S}$ would not be invariant. Since $\chi(k,t) = (\gamma(k), x_k(t))$, this translates to $x_k(t) \in \mathcal{S}_\lambda$. Whenever a discrete transition takes place (with regard to an edge $e \in \mathcal{E}$), the reset $\mathcal{R}(e)$ has to map the corresponding state again into $\mathcal{S}$ since otherwise invariance would be violated. Hence, we arrive at the following simple characterization based on own considerations.

3.5.4 Lemma. *Let $\mathscr{H}$ be a hybrid dynamical system. A set $\mathcal{S} \subset D$ is invariant if and only if it is locally invariant (in the classical sense), i.e. for every execution χ starting in $\mathcal{S}$ and every $\lambda \in \Lambda$*

$$x_{\gamma(k)}(t) \in \mathcal{S}_\lambda = \mathcal{S} \cap \mathcal{D}(\lambda) \qquad \textit{for all} \quad k \in \gamma^{-1}(\lambda) \quad \textit{and} \quad t \in I_k, \tag{3.111}$$

and reset-invariant, i.e.

$$\mathcal{R}(e)\,(\mathcal{G}(e) \cap \mathcal{S}_{\mathbb{T}}) \subset D_{\mathbb{T},\mathrm{te}} \cap \mathcal{S}_{\mathbb{T}} \quad \textit{for all } e \in \mathcal{E} \tag{3.112}$$

with $\mathcal{S}_{\mathbb{T}} = \mathcal{S} \times \mathbb{T}$ denoting the temporal extension of $\mathcal{S}$.

The discrete traits of hybrid invariance turn out to be connected with the notions of invariant subgraphs and output sets which are introduced and examined in the following.

3.5.5 Definition (Invariant Subgraph). Let $\mathscr{T} = (\Lambda, \mathcal{E})$ be a directed graph. A subgraph $\mathscr{T}' = (\Lambda', \mathcal{E}')$ of $\mathscr{T}$ is called *invariant* if every directed path $\omega : \mathbb{Z} \to \Lambda$ of $\mathscr{T}$ starting in $\mathscr{T}'$ stays there, i.e. $\omega_i = \omega(i) \in \Lambda'$ and $(\omega_i, \omega_{i+1}) \in \mathcal{E}'$ for all $i \in \mathbb{Z}$. ♢

3.5.6 Definition (Output Sets). Let $\mathscr{T} = (\Lambda, \mathcal{E})$ be a directed graph. The *$\mathscr{T}$-output set $\mathcal{O}_{\mathscr{T}}(\lambda)$* of a vertex $\lambda \in \Lambda$ is given by

$$\mathcal{O}_{\mathscr{T}}(\lambda) = \{e \in \mathcal{E} |\ \mathfrak{s}e = \lambda\} . \tag{3.113}$$

For a subset $L \subset \Lambda$ of vertices , we set

$$\mathcal{O}_{\mathscr{T}}(L) = \bigcup_{\lambda \in L} \mathcal{O}_{\mathscr{T}}(\lambda) = \{e \in \mathcal{E} |\ \mathfrak{s}e \in L\} \tag{3.114}$$

for the output set of L. ♢

We spotlight the relation of invariant subgraphs and output sets in the following lemma.

3.5.7 Lemma. *Let $\mathscr{T} = (\Lambda, \mathcal{E})$ be a directed graph and $\mathscr{T}' = (\Lambda', \mathcal{E}')$ a subgraph of $\mathscr{T}$. Then the following statements are equivalent:*

(i) *$\mathscr{T}'$ is invariant.*

(ii) *$\mathcal{O}_{\mathscr{T}}(\Lambda') \subset \mathcal{E}'$.*

Proof. $[(i) \Rightarrow (ii)]$ Let $\mathscr{T}'$ be invariant and let $e \in \mathcal{O}_{\mathscr{T}}(\lambda)$ for $\lambda \in \Lambda'$. Then there is a dipath ω of $\mathscr{T}$ starting in $\omega_0 = \lambda$ such that $(\omega_0, \omega_1) = e$. By invariance of $\mathscr{T}$, we know that $e \in \mathcal{E}'$. Since this argument holds for all $e \in \mathcal{O}_{\mathscr{T}}(\lambda)$, we see that $\mathcal{O}_{\mathscr{T}}(\lambda) \subset \mathcal{E}'$. Moreover, the inclusion holds for every $\lambda \in \Lambda'$ which leaves us to conclude that $\mathcal{O}_{\mathscr{T}}(\Lambda') \subset \mathcal{E}'$.

$[(ii) \Rightarrow (i)]$ Let ω a dipath of $\mathscr{T}$ starting in $\omega_0 = \lambda \in \Lambda'$. Then $e_{0,1} = (\omega_0, \omega_1) \in \mathcal{O}_{\mathscr{T}}(\lambda) \subset \mathcal{E}'$ by (ii). Inductively, $e_{i,i+1} = (\omega_i, \omega_{i+1}) \in \mathcal{O}(\omega_i) \subset \mathcal{E}'$ for all $i \in \mathbb{N}$ verifying invariance of $\mathscr{T}'$. ■

As indicated above by Lemma 3.5.2, a specific subgraph of the transition graph $\mathscr{T}$ plays a crucial role in describing the mixed structure of hybrid fixed-point spaces. This subgraph turns out to be the fixed graph with respect to a subgroup of $\mathrm{Aut}\,(\mathscr{T})$.

3.5.8 Definition. Let $\mathscr{T} = (\Lambda, \mathcal{E})$ be a directed graph. For a subgroup $\Gamma \leq \mathrm{Aut}\,(\mathscr{T})$, the Γ-*induced fixed graph* $\mathrm{Fix}_\Gamma(\mathscr{T}) = (\Lambda', \mathcal{E}') \sqsubseteq \mathscr{T}$ is determined by the vertices $\Lambda' = \mathrm{Fix}_\mathscr{T}(\Gamma)$ and the edges $\mathcal{E}' = (\Lambda' \times \Lambda') \cap \mathcal{E}$. $\diamondsuit$

We characterize the invariance of a Γ-induced fixed graph in terms of the Γ-action on the output set $\mathcal{O}(\mathrm{Fix}_\mathscr{T}(\Gamma))$.

3.5.9 Lemma. *Let $\mathscr{T} = (\Lambda, \mathcal{E})$ be a directed graph and $\mathrm{Fix}_\Gamma(\mathscr{T}) = (\Lambda', \mathcal{E}')$ its fixed graph induced by a subgroup $\Gamma \leq Aut(\mathscr{T})$. Then $\mathrm{Fix}_\Gamma(\mathscr{T})$ is invariant if and only if Γ fixes an edge $e \in \mathcal{E}$ of $\mathscr{T}$ whenever it fixes its source $\mathfrak{s}e$.*

Proof. By means of Lemma 3.5.7, invariance of $\mathrm{Fix}_\Gamma(\mathscr{T})$ is equivalent to the inclusion $\mathcal{O}_\mathscr{T}(\Lambda') \subset \mathcal{E}'$, where $\mathcal{O}_\mathscr{T}(\Lambda') = \{e \in \mathcal{E} |\ \mathfrak{s}e \in \Lambda'\}$. For an element $e \in \mathcal{O}_\mathscr{T}(\Lambda')$, one has $\pi^{-1}(\mathfrak{s}e) = \mathfrak{s}e$ for all $\pi \in \Gamma$ by definition and it is an edge of the fixed graph $\mathrm{Fix}_\Gamma(\mathscr{T})$ if and only if additionally $\pi^{-1}(\mathfrak{t}e) = \mathfrak{t}e$ holds for all $\pi \in \Gamma$, i.e. if $\pi^{-1}(e) = e$ for all $\pi \in \Gamma$. Consequently, $\mathcal{O}_\mathscr{T}(\Lambda') \subset \mathcal{E}'$ holds true iff Γ fixes every edge $e \in \mathcal{E}$ whose source $\mathfrak{s}e$ is fixed by Γ. ■

Taking into account the preceding considerations, we finally provide a sufficient condition for the invariance of the hybrid fixed-point space $\mathrm{Fix}(\Sigma)$ which essentially affects the discrete part $\widehat{\Sigma} \leq \mathrm{Aut}\,(\mathscr{T})$ of the group $\Sigma \leq \mathcal{H}$.

3.5.10 Proposition. *Let $\mathscr{H}$ be a hybrid dynamical system and $\Sigma \leq \mathcal{H}$ a subgroup of its hybrid symmetries. Moreover, let the resets $\mathcal{R}(e)$ be single-valued for $e \in \mathcal{O}_\mathscr{T}(\mathrm{Fix}_\mathscr{T}(\widehat{\Sigma}))$. If the $\widehat{\Sigma}$-induced fixed graph $\mathrm{Fix}_{\widehat{\Sigma}}(\mathscr{T})$ is invariant, the hybrid fixed-point space $\mathrm{Fix}(\Sigma)$ is invariant as well.*

Proof. Due to Lemma 3.5.4 it is sufficient to prove local flow-invariance and reset-invariance of $\mathrm{Fix}(\Sigma)$. For $p = (\lambda, x) \in \mathrm{Fix}(\Sigma)$ and $\Upsilon = (\pi, g) \in \Sigma$, Lemma 3.2.11 applies (since $\pi \in \Sigma_\lambda$ is equivalent to $\lambda \in \mathrm{Fix}_\mathscr{T}(\langle\pi\rangle) \subset \mathrm{Fix}_\mathscr{T}(\Sigma)$) and we have

$$g^{-1}_{\pi^{-1}(\lambda)} \mathcal{F}(\lambda)(x) = g_\lambda^{-1} \mathcal{F}(\lambda)(x) = \mathcal{F}(\lambda)\left(g_\lambda^{-1} x\right) = \mathcal{F}(\lambda)(x).$$

Thus by the classical result, $\mathrm{Fix}(\Sigma)$ is locally flow-invariant. Let $e \in \mathcal{O}_{\mathscr{T}}(\mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma}))$ be a discrete transition. For $p = (\mathfrak{s}e, x) \in \mathrm{Fix}(\Sigma) \cap \mathcal{G}(e)$ and $\Upsilon = (\pi, g) \in \Sigma$, we have

$$\mathcal{R}(e) = \Upsilon\mathcal{R}(e) = g_e^{-1}\mathcal{R}(\pi^{-1}(e)) = g_e^{-1}\mathcal{R}(e)$$

since we have $\pi^{-1}(e) = e$ due to Lemma 3.5.9. Now, g_e-invariance of the graph $\mathcal{R}(e) \subset D_{\mathbb{T},\mathfrak{s}e} \times D_{\mathbb{T},\mathfrak{t}e}$ enforces the equivariance $g_{\mathfrak{t}e}^{-1} \circ R_e = R_e \circ g_{\mathfrak{s}e}^{-1}$ of the underlying reset maps which is due to Lemma 3.3.2. Hence, we find

$$g_{\mathfrak{t}e}^{-1} R_e(x) = R_e\left(g_{\mathfrak{s}e}^{-1} x\right) = R_e(x) \tag{3.115}$$

for all $x \in \mathcal{G}(e) \cap \mathrm{Fix}(\Sigma)$. Since $\mathrm{card}(\mathcal{R}(e)(x)) = 1$, Equation (3.115) implies reset-invariance of $\mathrm{Fix}(\Sigma)$. Consequently, $\mathrm{Fix}(\Sigma)$ is invariant. ■

3.5.11 Example. Let us consider the hybrid dynamical system $\mathscr{H}_2$ displayed in Fig. 3.7 with vector fields $\mathcal{F}(1)$ and $\mathcal{F}(2)$ last structurally discussed in Example 3.2.14 and vector field $\mathcal{F}(3) : (\mathbb{R}^2)^4 \to (\mathbb{R}^2)^4$ defined by

$$\mathcal{F}(3)(z) = \begin{pmatrix} A & B \\ B & A \end{pmatrix} z + H(z) \tag{3.116}$$

with

$$A = \begin{pmatrix} C_2 & \zeta\mathbb{I}_2 \\ \zeta\mathbb{I}_2 & C_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \zeta\mathbb{I}_2 \\ \zeta\mathbb{I}_2 & 0 \end{pmatrix} \tag{3.117}$$

and C_2 as in (3.58). This special structure imposes D_4-equivariance on $\mathcal{F}(3)$, i.e. we have $S_3 = D_4 \leq S_4$. We want to figure out the hybrid symmetries $\mathcal{H}_2$ of $\mathscr{H}_2$. First of all, the transition graph $\mathscr{T}$ has the automorphism group $\mathrm{Aut}(\mathscr{T}) = \langle (12) \rangle \cong \mathbb{Z}_2$. The $\mathscr{T}$-symmetries of the underlying $\mathscr{T}$-system are captured by $\mathfrak{S}_2 = \mathfrak{S} \times D_4$ where $\mathfrak{S} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is the $\mathscr{T}$-symmetry group of the $\mathscr{T}$-system treated in Example 3.2.14. This is true because the discrete state 3 is fixed by $\mathrm{Aut}(\mathscr{T})$; apart from that the vector field $\mathcal{F}(3)$ is not related to the other two vector fields so that $\mathscr{T}$-symmetries involving $\mathcal{F}(3)$ in a connective manner cannot arise anyhow.

Now we address the guards. For $\lambda \in \Lambda$, a subgroup $H_\lambda \leq G_\lambda$ and $\varepsilon > 0$ the ε-approximated fixed-point space $\mathrm{Fix}_\varepsilon(H_\lambda)$ is given by

$$\mathrm{Fix}_\varepsilon(H_\lambda) = \left\{(\lambda, x) \in \mathcal{D}(\lambda) \mid \mathrm{dist}_{\|\cdot\|}\left(x, \mathrm{Fix}(H_\lambda)\right) < \varepsilon\right\}. \tag{3.118}$$

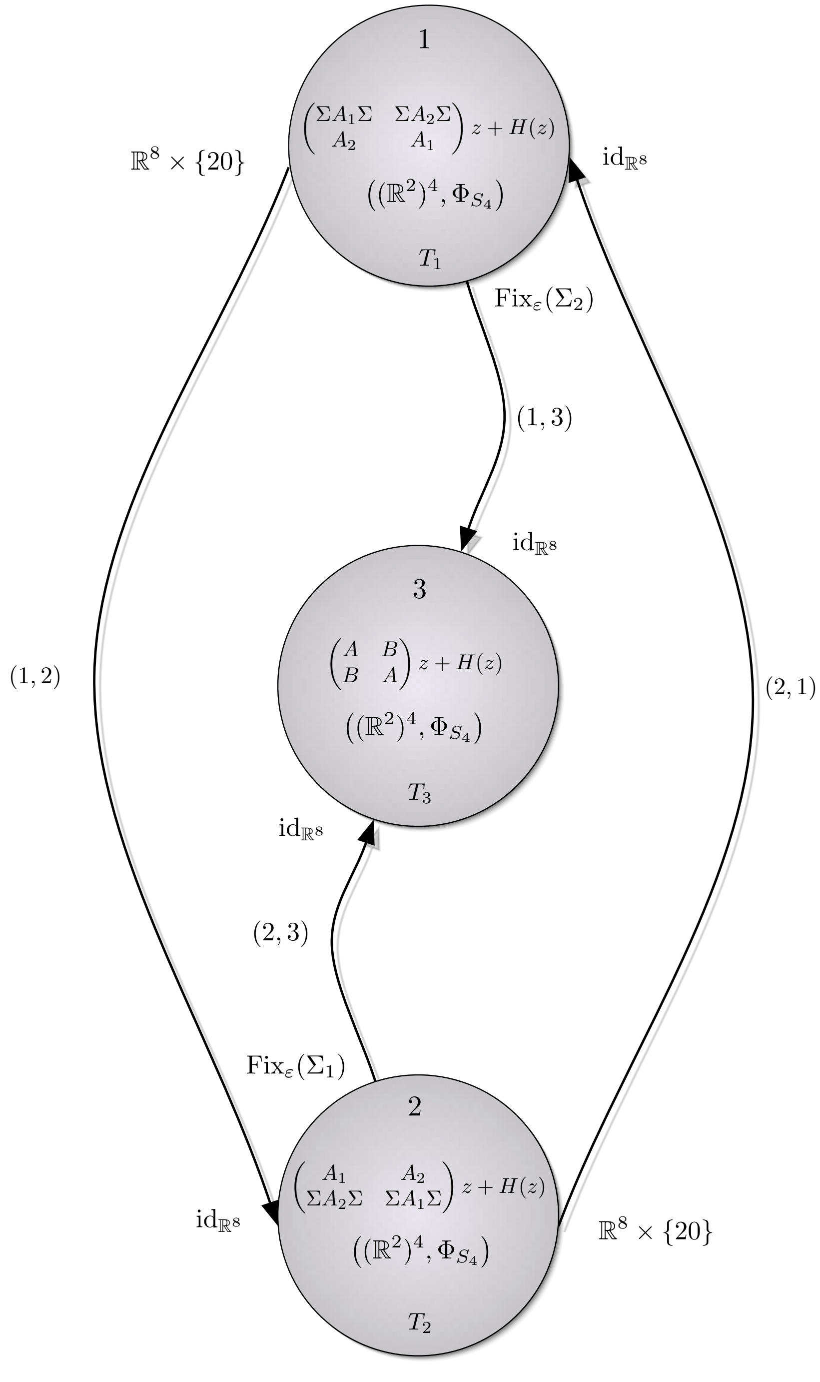


Figure 3.7: Hybrid dynamical system $\mathscr{H}_2$ with hybrid symmetries $\mathcal{H}_2 \cong \mathbb{Z}_2 \times D_4$ and invariant fixed-point spaces $\mathrm{Fix}(\Xi)$ for each subgroup $\Xi \leq \mathcal{H}_2$.

Recall that the classical fixed-point space $\mathrm{Fix}(H_\lambda)$ is $N_{G_\lambda}(H_\lambda)$-invariant. Choosing an $N_{G_\lambda}(H_\lambda)$-invariant norm $||\cdot||$, we compute for $(\lambda, x) \in \mathrm{Fix}_\varepsilon(H_\lambda)$ and $g \in N_{G_\lambda}(H_\lambda)$

$$\begin{aligned}
\mathrm{dist}_{||\cdot||}(gx, \mathrm{Fix}(H_\lambda)) &= \mathrm{dist}_{||\cdot||}(gx, g\mathrm{Fix}(H_\lambda)) \\
&= \min_{y \in \mathrm{Fix}(H_\lambda)} d_{||\cdot||}(gx, gy) \\
&= \min_{y \in \mathrm{Fix}(H_\lambda)} d_{||\cdot||}(x, y) \\
&= \mathrm{dist}_{||\cdot||}(x, \mathrm{Fix}(H_\lambda)) < \varepsilon.
\end{aligned}$$

Thus we find $gx \in \mathrm{Fix}_\varepsilon(H_\lambda)$ and consequently $g\mathrm{Fix}_\varepsilon(H_\lambda) \subset \mathrm{Fix}_\varepsilon(H_\lambda)$. Also, we find $g\mathrm{Fix}_\varepsilon(H_\lambda) \subset \mathrm{Fix}_\varepsilon(H_\lambda)$, finally implying $g\mathrm{Fix}_\varepsilon(H_\lambda) = \mathrm{Fix}_\varepsilon(H_\lambda)$. Hence $\mathrm{Fix}_\varepsilon(H_\lambda)$ is $N_{G_\lambda}(H_\lambda)$-invariant.

Let the subgroups $\Sigma_1, \Sigma_2 \leq S_4$ be given by

$$\Sigma_1 = \langle(24)\rangle \quad \text{and} \quad \Sigma_2 = \langle(13)\rangle. \tag{3.119}$$

With guards

$$\mathcal{G}(1,3) = \mathrm{Fix}_\varepsilon(\Sigma_2) \quad \text{and} \quad \mathcal{G}(2,3) = \mathrm{Fix}_\varepsilon(\Sigma_1), \tag{3.120}$$

the guard stabilizer $\Sigma_\mathcal{G}$ becomes

$$\Sigma_\mathcal{G} = \{(\mathrm{id}_\Lambda, \mathrm{id}, \mathrm{id}, g_3), (\pi, (12)(34), (12)(34), g_3) \mid g_3 \in D_4\}, \tag{3.121}$$

and thus we obtain the hybrid symmetry group

$$\mathcal{H}_2 = \Sigma_\mathcal{G} \cong \mathbb{Z}_2 \times D_4 \tag{3.122}$$

for $\mathscr{H}_2$. Let Ξ be a subgroup of $\mathcal{H}_2$ with $\widehat{\Xi} \cong \mathbb{Z}_2$. We aim to examine $\mathrm{Fix}(\Xi)$. First, we identify the $\widehat{\Xi}$-induced fixed graph $\mathrm{Fix}_{\widehat{\Xi}}(\mathscr{T})$: Its vertices are given by $\mathrm{Fix}_\mathscr{T}(\widehat{\Xi}) = \{3\}$ and the edges $\mathcal{E}' = \emptyset$ since $(3,3)$ is not an edge of $\mathscr{T}$. Thus $\mathrm{Fix}_{\widehat{\Xi}}(\mathscr{T})$ is the trivial subgraph of $\mathscr{T}$ solely consisting of the discrete state 3. Recall that $\mathscr{T}$ has edges $\mathcal{E} = \{(1,2), (2,1), (1,3), (2,3)\}$, i.e. there is no $e \in \mathcal{E}$ whose source is fixed by $\widehat{\Xi}$. In this situation, Lemma 3.5.9 yields invariance of the fixed graph $\mathrm{Fix}_{\widehat{\Xi}}(\mathscr{T})$ (in fact, this is obvious since its output set is empty). By means of Proposition 3.5.10, $\mathrm{Fix}(\Xi)$ is found to be invariant. $\diamondsuit$

By an explicit incorporation of the guards as triggering instances for discrete transitions, Proposition 3.5.10 can be generalized as follows.

3.5.12 Theorem. *Let $\mathscr{H}$ be a hybrid dynamical system with single-valued resets $\mathcal{R}(e)$ for $e \in \mathcal{O}_{\mathscr{T}}(\mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma}))$ and let $\Sigma \leq \mathcal{H}$ be a subgroup of its hybrid symmetries such that for each edge $e \in \mathcal{O}_{\mathscr{T}}(\mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma}))$ at least one of the following two statements holds:*

(1) $\mathfrak{t}e \in \mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma})$

(2) $\mathrm{Fix}_{\mathfrak{s}e}(\widehat{\Sigma}_{\mathfrak{s}e}) \cap \mathcal{G}(e) = \emptyset$.

Then $\mathrm{Fix}(\Sigma)$ *is invariant.*

Proof. Let χ be an execution of $\mathscr{H}$ starting in $p_0 = (\lambda_0, x_0) \in \mathrm{Fix}\,(\Sigma)$. Consider an edge $e \in \mathcal{O}_{\mathscr{T}}(\mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma}))$ of $\mathscr{T}$. By definition, the source $\mathfrak{s}e$ of e is a vertex of the fixed graph $\mathrm{Fix}_{\widehat{\Sigma}}(\mathscr{T})$, meaning $\pi^{-1}(\mathfrak{s}e) = \mathfrak{s}e$ for all $\pi \in \widehat{\Sigma}$. If additionally its target $\mathfrak{t}e$ is an element of $\mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma})$, then e itself is fixed by $\widehat{\Sigma}$ and an execution visiting $D_{\mathfrak{s}e}$ is unable to quit the vertex set $\Lambda' = \mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma})$ via e since it will be reset to an element of Λ' by $\mathcal{R}(e)$. So is χ, and as in the proof of Proposition 3.5.10 by virtue of the hybrid symmetries, χ is reset to the local fixed-point space $\mathrm{Fix}_{\mathfrak{t}e}(\widehat{\Sigma}_{\mathfrak{t}e})$ by $\mathcal{R}(e)$ (cp. Eq. (3.115)).

In case there is $\pi \in \Sigma$ with $\pi^{-1}(\mathfrak{t}e) \neq \mathfrak{t}e$ and the isolation condition $\mathrm{Fix}_{\mathfrak{s}e}(\widehat{\Sigma}_{\mathfrak{s}e}) \cap \mathcal{G}(e) = \emptyset$ holds, an execution staying in $\mathrm{Fix}_{\mathfrak{s}e}(\widehat{\Sigma}_{\mathfrak{s}e})$ cannot leave Λ' by e either since it cannot hit the guard $\mathcal{G}(e)$ triggering a transition along e.

Hence, in both cases the execution χ cannot leave the fixed-point set $\mathrm{Fix}(\Sigma)$ showing that $\mathrm{Fix}\,(\Sigma)$ is invariant. ∎

Obviously, if condition (1) holds for all edges in question, we find ourselves in the situation of Proposition 3.5.10 since $\mathfrak{t}e \in \mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma})$ for every edge $e \in \mathcal{O}_{\mathscr{T}}(\mathrm{Fix}_{\mathscr{T}}(\widehat{\Sigma}))$ is equivalent to the invariance of the $\widehat{\Sigma}$-induced fixed graph $\mathrm{Fix}_{\widehat{\Sigma}}(\mathscr{T})$ by Lemma 3.5.9. In case of purely temporal switching, condition (2) cannot be met non-trivially at all since spatially the guards are then given by the complete space $\mathcal{D}(\lambda)$ which has an empty intersection with the $\mathfrak{s}e$-local fixed-point space $\mathrm{Fix}_{\mathfrak{s}e}(\widehat{\Sigma}_{\mathfrak{s}e})$ if and only if $\mathrm{Fix}_{\mathfrak{s}e}(\widehat{\Sigma}_{\mathfrak{s}e})$ is empty itself.

3.5.13 Example. We consider the hybrid dynamical system $\mathscr{H}_3$ illustrated in Fig. 3.8. The dynamical systems Ψ_1, Ψ_2 and Ψ_3 are given as for $\mathscr{H}_2$ in Example 3.5.11. Note that $\mathscr{H}_3$ is similar to $\mathscr{H}_2$; the only difference lies in the additional edges $(3,1)$ and $(3,2)$ with according guard and reset data.

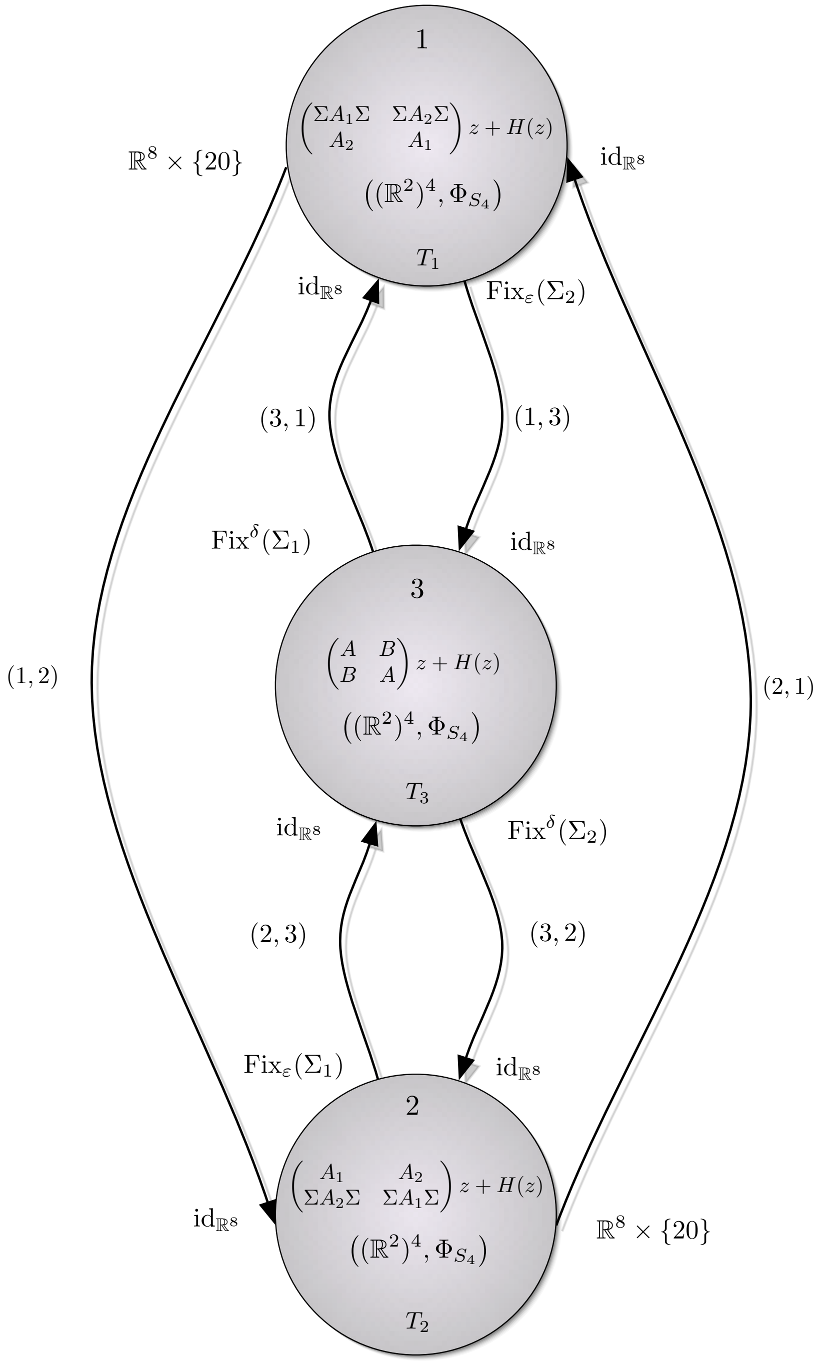


Figure 3.8: Hybrid dynamical system $\mathscr{H}_3$ with hybrid symmetries $\mathcal{H}_3 \cong \mathbb{Z}_2 \times D_4$ and invariant hybrid fixed-point space $\mathrm{Fix}(\Xi)$ with $\Xi = \langle(\pi, (12)(34), (12)(34), (12)(34))\rangle \cong \mathbb{Z}_2$.

For $\lambda \in \Lambda$, $H_\lambda \leq G_\lambda$ and $\delta > 0$, we set

$$\mathrm{Fix}^\delta(H_\lambda) = \left\{(\lambda, x) \in \mathcal{D}(\lambda) \mid \mathrm{dist}_{||\cdot||}(x, \mathrm{Fix}(H_\lambda)) > \delta\right\}. \tag{3.123}$$

Analogous to $\mathrm{Fix}_\varepsilon(H_\lambda)$, $\mathrm{Fix}^\delta(H_\lambda)$ is $N_{G_\lambda}(H_\lambda)$-invariant. Let

$$\mathcal{G}(3,1) = \mathrm{Fix}^\delta(\Sigma_1) \quad \text{and} \quad \mathcal{G}(3,2) = \mathrm{Fix}^\delta(\Sigma_2) \tag{3.124}$$

be the guards for the two recent edges with $\Sigma_1 = \langle(24)\rangle$ and $\Sigma_2 = \langle(13)\rangle$. We assume that $\delta \geq \varepsilon$. Observe that the $\mathscr{T}$-symmetry group as well as the guard stabilizer are the same as in Example 3.2.14. Therefore, we also end up with the same hybrid symmetries $\mathcal{H}_3 \cong \mathbb{Z}_2 \times D_4$. We specify the subgroup $\Xi \leq \mathcal{H}_3$ with $\widehat{\Xi} \cong \mathbb{Z}_2$ by

$$\Xi = \langle(\pi, (12)(34), (12)(34), (12)(34))\rangle \cong \mathbb{Z}_2. \tag{3.125}$$

In this example the $\widehat{\Xi}$-induced fixed graph $\mathrm{Fix}_{\widehat{\Xi}}(\mathscr{T})$ is still given by the simple vertex 3. However, in this case the output set of the discrete state 3 is not-empty since we have

$$\mathcal{O}_{\mathscr{T}}\left(\mathrm{Fix}_{\mathscr{T}}(\widehat{\Xi})\right) = \{(3,1),(3,2)\}$$

implying that the fixed graph $\mathrm{Fix}_{\widehat{\Xi}}(\mathscr{T})$ is not invariant. This is why Proposition 3.5.10 does not apply. For the 3-localized fixed-point space $\mathrm{Fix}_3(\widehat{\Xi}_3)$, we obtain

$$\mathrm{Fix}_3(\widehat{\Xi}_3) = \mathrm{Fix}_3\left(\langle(12)(34)\rangle\right) \tag{3.126}$$

and observe that $\mathrm{Fix}\left(\langle(12)(34)\rangle\right) \subset \mathrm{Fix}\left(\langle(12)\rangle\right)$ and $\mathrm{Fix}\left(\langle(12)(34)\rangle\right) \subset \mathrm{Fix}\left(\langle(34)\rangle\right)$. Futhermore, on account of the relation $\delta \geq \varepsilon$ we note that $\mathrm{Fix}^\delta(\Sigma_i) \subset \{3\} \times \mathbb{R}^8 \setminus \mathrm{Fix}_\varepsilon(\Sigma_i)$ implying

$$\mathcal{G}(3,1) \cap \mathrm{Fix}_3(\widehat{\Xi}_3) = \emptyset \quad \text{and} \quad \mathcal{G}(3,2) \cap \mathrm{Fix}_3(\widehat{\Xi}_3) = \emptyset.$$

Hence, our example meets the assumptions of Theorem 3.5.12 which in turn ensures the invariance of $\mathrm{Fix}(\Xi)$. $\diamondsuit$

Chapter 4

Periodicity in Hybrid Dynamics and Hybrid Spatio-Temporal Symmetries

Contrary to the *spatial* symmetry a solution $x(t)$ of a G-equivariant dynamical system Ψ can exhibit which is encoded by its isotropy subgroup $\Sigma_{x(t)} \leq G$, it may possess symmetry in *time*. The simplest (and purest) case of temporal symmetry is periodicity of $x(t)$, i.e. there exists $T > 0$ such that $x(t+T) = x(t)$ for all $t \in \mathbb{R}$. If there exists a period T which is minimal with respect to this property, the group of temporal symmetries of $x(t)$ is given by the subgroup $T\mathbb{Z} \leq \mathbb{R}$. T-periodicity can also be formulated as an invariance property with regard to the shift action $\sigma_T x(\cdot) = x(\cdot + T)$, where the shift operator σ_T acts on maps $C(\mathbb{R}, \mathbb{R}^n)$ rather than on vectors. Spatial and temporal symmetries may be viewed as the boundary-posts of symmetry with regard to the solutions of dynamical systems that give rise to an *intermingled* notion of symmetry – termed *spatio-temporal symmetry* – in between, which are examined in [Fie88] and [GS02], for instance. The occurence of spatio-temporally symmetric phenomena is not restricted to single solutions of autonomous dynamical systems, but may more generally be treated for non-autonomous systems that are forced periodically in time, i.e. for the explicitely time-dependent vector field $F : \mathbb{R} \times D \to \mathbb{R}^n$, a period $T > 0$ exists such that $F(t + T, \cdot) = F(t, \cdot)$ for all $t \in \mathbb{R}$ (see [Lam98]).

After hybrid symmetries have been set up successfully in the last chapter, this chapter intends to trace the classical idea of spatio-temporal symmetries in the framework of hybrid dynamical systems. Subsequently, we describe and inspect the periodicity of executions in Section 4.1 whereupon hybrid spatio-temporal symmetries that conceptually build on Chapter 3 are treated in Section 4.2. Finally, in Section 4.3, we will turn away from specific executions and discuss hybrid spatio-temporal symmetry properties in a broader sense, namely of hybrid dynamical systems with respect to a given periodic switching signal. In this context, we figure out the structural consequences of hybrid spatio-temporal symmetries on the hybrid time-T map over one period of the switching signal in question.

4.1 Periodic Executions

In the course of this section we aim to explore periodicity – or purely temporal symmetry in a manner of speaking – in the framework of hybrid dynamical systems. By means of an execution we unfold the notion of hybrid periodicity and work out the hybrid details setting it apart from the periodicity of classical trajectories.

4.1.1 Definition (Hybrid Periodicity). An execution $\chi : \mathcal{T} \to D$ of a hybrid dynamical system $\mathscr{H}$ is *periodic* if there exists a pair $P_\chi = (N_\chi, T_\chi) \in \mathbb{N} \times \mathbb{R}_{\geq 0}$ such that

$$\chi(\phi + P_\chi) = \chi(\phi) \tag{4.1}$$

holds for all $\phi = (k,t) \in \mathcal{T}$. P_χ is called the *hybrid period* of χ with N_χ being the *discrete* and T_χ the *continuous* period of χ. $\diamondsuit$

Recently, the equivalent notion of *hybrid periodic orbit* has explicitely occured in the article [WA10], where rank properties of Poincaré maps are investigated. Hybrid periodicity can be straightforwardly characterized in terms of discrete and continuous components as follows.

4.1.2 Lemma. *An execution χ is P_χ-periodic with $P_\chi = (N_\chi, T_\chi)$ if and only if the relations*

$$\gamma(k + N_\chi) = \gamma(k) \quad \textit{and} \quad x_{k+N_\chi}(t + T_\chi) = x_k(t) \tag{4.2}$$

hold for all $(k,t) \in \mathcal{T}$.

Proof. Decryption of (4.1) via (2.14) yields the desired statement. ∎

It is important to note that for an execution χ with minimal period $P_\chi = (N_\chi, T_\chi)$, the discrete part N_χ is not necessarily minimal when considered as the discrete period of γ, i.e. there generally exists a minimal $N < N_\chi$ such that $\gamma(k+N) = \gamma(k)$ for all k. Obviously, N has to divide N_χ.

An execution χ determines a subgraph $\mathscr{T}_\chi$ of the transition graph $\mathscr{T} = (\Lambda, \mathcal{E})$ in the following way, which I have not encountered in the literature so far.

4.1.3 Definition. For an execution χ of $\mathscr{H}$, the χ-*induced transition graph* $\mathscr{T}_\chi = (\Lambda_\chi, \mathcal{E}_\chi)$ is determined by

$$\Lambda_\chi = \mathrm{im}(\gamma) \subset \Lambda \quad \text{and} \quad \mathcal{E}_\chi = \{(\gamma(k), \gamma(k+1)) \mid k \in [\tau]\} \subset \mathcal{E}. \tag{4.3}$$

Its vertices are given by the image of the discrete state map γ and its edges are given by adjacent values of γ. Descriptively speaking, the graph $\mathscr{T}_\chi$ traces the path an execution χ takes inside the transition graph $\mathscr{T}$ without unfolding it by marking the part of $\mathscr{T}$ the execution visits along its way. This subgraph will turn out to be useful in the course of the forthcoming analysis.

Below, we mention the implications of periodicity on the χ-induced transition graph $\mathscr{T}_\chi$ (for the χ-induced transition graph see Definition 4.1.3).

4.1.4 Corollary. *Let χ be a periodic execution with minimal period $P_\chi = (N_\chi, T_\chi)$. Then if $N_\chi \neq 0$, the χ-induced subgraph $\mathscr{T}_\chi = (\Lambda_\chi, \mathcal{E}_\chi)$ is a cycle graph with* $\mathrm{card}(\Lambda_\chi) \mid N_\chi$.

Proof. By Lemma 4.1.2, one has $\gamma(k + N_\chi) = \gamma(k)$ for all $k \in \mathbb{Z}_{\geq 0}$. Since $(\gamma(k+N-1), \gamma(k+N)) = (\gamma(k+N-1), \gamma(k)) \in \mathcal{E}_\chi$, the tuple $(\gamma(0), \ldots, \gamma(N-1))$ represents a cycle of $\mathscr{T}_\chi$ and thus of $\mathscr{T}$. Notably, the periodicity of γ forces $\mathrm{card}(\Lambda_\chi) | N_\chi$. ∎

Note carefully that under the assumptions of the above corollary $\mathscr{T}_\chi$ is a *cycle* graph, but not necessarily a *cyclic* graph, since γ may run around in $\mathscr{T}$ arbitrarily. In case $N_\chi = 0$, the execution stays in a single discrete state all the time (forcing the induced transition graph to be trivial) and thus hybrid periodicity reduces to classical periodicity of a solution $x(t)$. In order to obtain a more detailed view on hybrid periodicity, we introduce the *discrete trace* of an execution χ which provides a graphical characterization of the discrete period.

4.1.5 Definition. Let χ be a periodic execution with period $P_\chi = (N_\chi, T_\chi)$. The *discrete trace* $\mathscr{T}^*_\chi = (\Lambda^*_\chi, \mathcal{E}^*_\chi)$ of χ is determined by the vertices

$$\Lambda^*_\chi = \left\{ \Big([k]_{N_\chi}, \gamma(k) \Big) \mid k \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Z}/N_\chi\mathbb{Z} \times \Lambda \tag{4.4}$$

and by

$$\mathcal{E}^*_\chi = \left\{ \Big(\Big([k]_{N_\chi}, \gamma(k) \Big), \Big([k+1]_{N_\chi}, \gamma(k+1) \Big) \Big) \Big| \; k \in \mathbb{Z}_{\geq 0} \right\} \subset \Lambda^*_\chi \times \Lambda^*_\chi \tag{4.5}$$

as edges. $\diamondsuit$

Note that $\mathscr{T}^*_\chi$ is a unidirectional cyclic graph of size N_χ that can be considered to be a refinement of $\mathscr{T}_\chi$ insofar as $\mathscr{T}^*_\chi$ untangles the χ-induced transition graph $\mathscr{T}_\chi$ by comprising the visiting multiplicities of discrete states. For a periodic execution χ, Figure 4.1 contrasts the χ-induced transition graph $\mathscr{T}_\chi$ with the according discrete trace $\mathscr{T}^*_\chi$ in an exemplary manner; in turn, this gives a discrete characterization of the execution displayed in Figure 4.2. Analogous to the χ-induced hybrid dynamical system $\mathscr{H}_\chi$ modelled on the χ-induced transition graph, we consider the *hybrid trace* $\mathscr{H}^*_\chi$ of χ to be the according hybrid dynamical system that is built on the discrete trace $\mathscr{T}^*_\chi$ of χ. More precisely, the hybrid trace $\mathscr{H}^*_\chi$ is given by the data $\left(\Lambda^*_\chi, \mathcal{E}^*_\chi, \Theta^*_\chi, \mathcal{F}^*_\chi, \mathcal{T}^*_\chi, \mathcal{G}^*_\chi, \mathcal{R}^*_\chi\right)$ with

$$\begin{aligned}
\Theta^*_\chi(\lambda_k) &= \Theta(\gamma(k)), \\
\mathcal{F}^*_\chi(\lambda_k) &= \mathcal{F}(\gamma(k)), \\
\mathcal{T}^*_\chi(\lambda_k) &= \mathcal{T}(\gamma(k)), \\
\mathcal{G}^*_\chi(e_k) &= \mathcal{G}(\gamma(k), \gamma(k+1)), \\
\mathcal{R}^*_\chi(e_k) &= \mathcal{R}(\gamma(k), \gamma(k+1)),
\end{aligned}$$

where $\lambda_k = ([k], \gamma(k))$ and $e_k = (\lambda_k, \lambda_{k+1})$.

Apparently, the discrete period N_χ and the continuous period T_χ are not at all separated quantities, but strongly interwoven. Therefore, we aim to characterize the hybrid period P_χ in more detail by examining the interplay between the

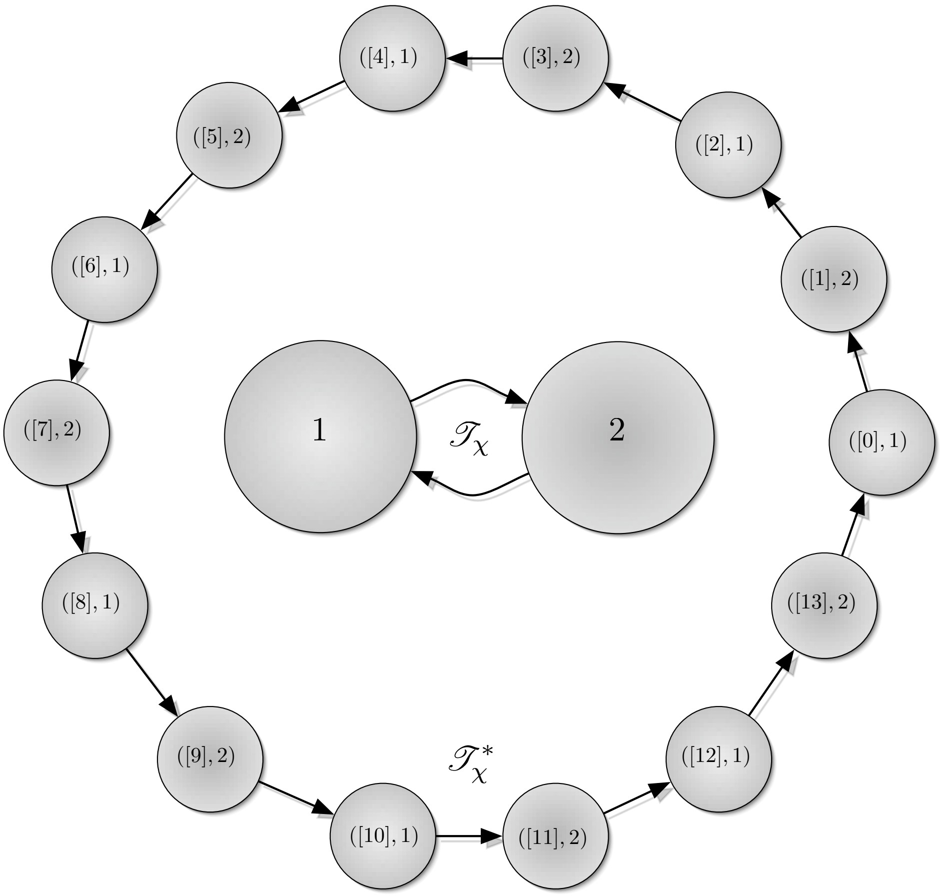


Figure 4.1: Induced transition graph $\mathscr{T}_\chi$ (center) and discrete trace $\mathscr{T}_\chi^*$ of a periodic execution χ with discrete period $N_\chi = 14$; cp. Figure 4.2.

discrete and the continuous period. For this purpose, we consider the following map based on the hybrid time set τ of χ:

$$\Theta_\tau^{(q)} : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad N \mapsto \sum_{k=q}^{q+N-1} |I_k|, \qquad q \in \mathbb{Z}_{\geq 0} \tag{4.6}$$

and set $\Theta_\tau = \Theta_\tau^{(0)}$. Note that for $N \in \mathbb{Z}_{\geq 0}$ and an n-tuple $(m_1, \ldots, m_n)$, $0 \leq n < N$, of integers satisfying $0 = m_0 < m_1 < \cdots < m_n < m_{n+1} = N$, one has

$$\Theta_\tau(N) = \sum_{j=0}^{n} \Theta_\tau^{(m_j)}(m_{j+1} - m_j). \tag{4.7}$$

It turns out that Θ_τ fundamentally relates the discrete to the continuous period.

4.1.6 Proposition. *Let $\chi = (\tau, \gamma, x)$ be a periodic execution of the hybrid dynamical system $\mathscr{H}$ with hybrid period $P_\chi = (N_\chi, T_\chi)$. Then $\Theta_\tau^{(q)}(N_\chi) = T_\chi$ for all $q \in \mathbb{Z}_{\geq 0}$ and Θ_τ induces a map*

$$\widetilde{\Theta}_\chi : \mathbb{Z}/N_\chi\mathbb{Z} \to \mathbb{R}/T_\chi\mathbb{Z}. \tag{4.8}$$

Proof. When considering χ as a map $\chi : \mathcal{T} \to D$, we observe that for $N \in \mathbb{Z}_{\geq 0}$ the time $T = \Theta_\tau(N)$ is the time the initial state $\chi(0,0)$ needs to evolve to $\chi(N, \tau_N)$ along χ, i.e. $\tau_N = T$. Hence, if χ is periodic with period $P_\chi = (N_\chi, T_\chi)$, we can conclude that $\Theta_\tau(N_\chi) = T_\chi$. The identity $x_{k+N_\chi}(t + T_\chi) = x_k(t)$ on I_k entails the relation $I_{k+N_\chi} = I_k + T_\chi$ with $I_k + T_\chi$ denoting the right-shift $[\tau_k + T_\chi, \tau_k' + T_\chi]$ of the interval I_k by T_χ. Thus,

$$\left|I_{k+N_\chi}\right| = |I_k| \qquad \text{for all } k \in \mathbb{Z}_{\geq 0} \tag{4.9}$$

and

$$\Theta_\tau(N_\chi) = \sum_{k=0}^{N_\chi - 1} |I_k| = \sum_{k=q}^{q+N_\chi-1} |I_k| = \Theta_\tau^{(q)}(N_\chi) \qquad \text{for all } q \in \mathbb{Z}_{\geq 0}. \tag{4.10}$$

Since every integer multiple of N_χ occurs as a discrete period of χ, as well, we have

$$\Theta_\tau^{(q)}(rN_\chi) = \Theta_\tau(rN_\chi) \overset{(4.7)}{=} \sum_{j=0}^{r-1} \Theta_\tau^{(jN_\chi)}(N_\chi) = \sum_{j=0}^{r-1} \Theta_\tau(N_\chi) = r\Theta_\tau(N_\chi) \tag{4.11}$$

for $r \in \mathbb{N}$ and $q \in \mathbb{Z}_{\geq 0}$, where we use the $(r-1)$-tuple $(m_1, \ldots, m_{r-1})$ given by $m_j = jN_\chi$. For $q \in \{0, 1, \ldots, N_\chi - 1\}$ we obtain

$$\begin{aligned}\Theta_\tau(q + rN_\chi) &= \Theta_\tau(q) + \Theta_\tau^{(q)}(rN_\chi) \\ &\overset{(4.11)}{=} \Theta_\tau(q) + r\Theta_\tau(N_\chi).\end{aligned}$$

Thence, the map

$$\widetilde{\Theta}_\chi : \mathbb{Z}/N_\chi\mathbb{Z} \to \mathbb{R}/T_\chi\mathbb{Z}, \quad \widetilde{\Theta}_\chi([N]) = [\Theta_\tau(N)]$$

is well-defined. ■

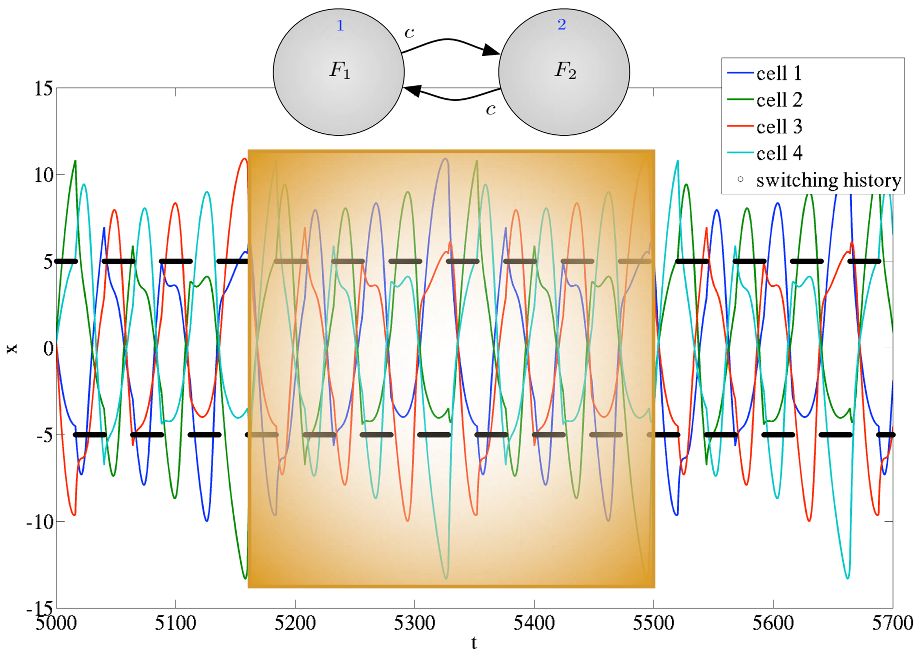


Figure 4.2: Execution χ with minimal hybrid period $P_\chi = (14, 336)$ and induced hybrid subsystem $\mathscr{H}_\chi$ displayed above

In case, χ is a periodic execution with *uniform* hybrid time trajectory, i.e. $|I_k| = \beta \in \mathbb{R}_{\geq 0}$ for all $k \in \mathbb{Z}$, one has

$$T_\chi = \Theta_\tau(N_\chi) = \beta N_\chi. \tag{4.12}$$

We say that χ is *temporally uniform.* Figure 4.2 shows a temporally uniform execution with period $P_\chi = (14, 336)$. By Eq. (4.12), we find that $\beta = |I_k| = 24$ in this example.

4.2 Hybrid Spatio-Temporal Symmetries

Having inspected hybrid periodicity, we aim to set up the concept of hybrid spatio-temporal symmetries for executions of hybrid dynamical systems guided by the classical notion of spatio-temporal symmetries for solutions of dynamical systems. From now on, we require that the hybrid dynamical systems we deal with are both *deterministic* and *non-blocking* (see Chapter 2.2).

Let $\Delta(\mathbb{Z}\times\mathbb{Z})$ denote the diagonal subgroup of $\mathbb{Z}\times\mathbb{Z}$. For a periodic execution χ of a hybrid dynamical system $\mathscr{H}$ with hybrid period $P_\chi = (N_\chi, T_\chi) \in \mathbb{Z}\times\mathbb{R}$, we note that $P_\chi\Delta(\mathbb{Z}\times\mathbb{Z}) = \{(kN_\chi, kT_\chi) \mid k \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}\times\mathbb{R}$ and consider the quotient group

$$\Pi_\chi = (\mathbb{Z}\times\mathbb{R})\,/P_\chi\Delta(\mathbb{Z}\times\mathbb{Z}). \tag{4.13}$$

Obviously, Π_χ may be perceived as a subgroup of $(\mathbb{Z}/N_\chi\mathbb{Z}) \times (\mathbb{R}/T_\chi\mathbb{Z})$.

We motivate the following definition of hybrid spatio-temporal symmetries in strong analogy to the classical case which is treated in [GS02] and [Fie88], for instance. In general, spatio-temporal symmetries arise when (spatial) symmetries meet (temporal) periodicity. In that spirit, we consider a hybrid dynamical system $\mathscr{H}$ with hybrid symmetries $\mathcal{H}$. Let χ be a periodic execution of $\mathscr{H}$ with hybrid period P_χ. Then by Proposition 3.4.2 for a hybrid symmetry $\Upsilon \in \mathcal{H}$, $\Upsilon\chi$ is an execution of $\mathscr{H}$, as well. Moreover, $\Upsilon\chi$ is periodic with the same period P_χ. There are two distinct cases that may occur: The images $\mathcal{I}_\chi$ and $\mathcal{I}_{\Upsilon\chi}$ of χ and $\Upsilon\chi$, respectively, intersect each other or they do not. In case $\mathcal{I}_\chi \cap \mathcal{I}_{\Upsilon\chi} \neq \emptyset$, the determinism of $\mathscr{H}$ forces $\mathcal{I}_\chi$ and $\mathcal{I}_{\Upsilon\chi}$ to coincide. Then, uniqueness of executions implies the existence of $P = (N, T) \in \mathbb{Z}\times\mathbb{R}$ with $0 \le N \le N_\chi$ and $0 \le T \le T_\chi$ such that $\Upsilon\chi(\phi - P) = \chi(\phi)$ holds for all $\phi \in \mathcal{T}$. Upon this observation, we formulate the subsequent definition.

4.2.1 Definition (Spatio-Temporal Symmetries). Let $\mathscr{H}$ be a hybrid dynamical system with hybrid symmetry group $\mathcal{H}$ and χ a periodic execution of

$\mathscr{H}$ with hybrid period $P_\chi = (N_\chi, T_\chi)$. An element $(\Upsilon, P) \in \mathcal{H} \times \Pi_\chi$ is a *hybrid spatio-temporal symmetry* of χ if

$$\Upsilon\chi(\phi - P) = \chi(\phi) \tag{4.14}$$

holds for all $\phi = (k,t) \in \mathcal{T}$. ◊

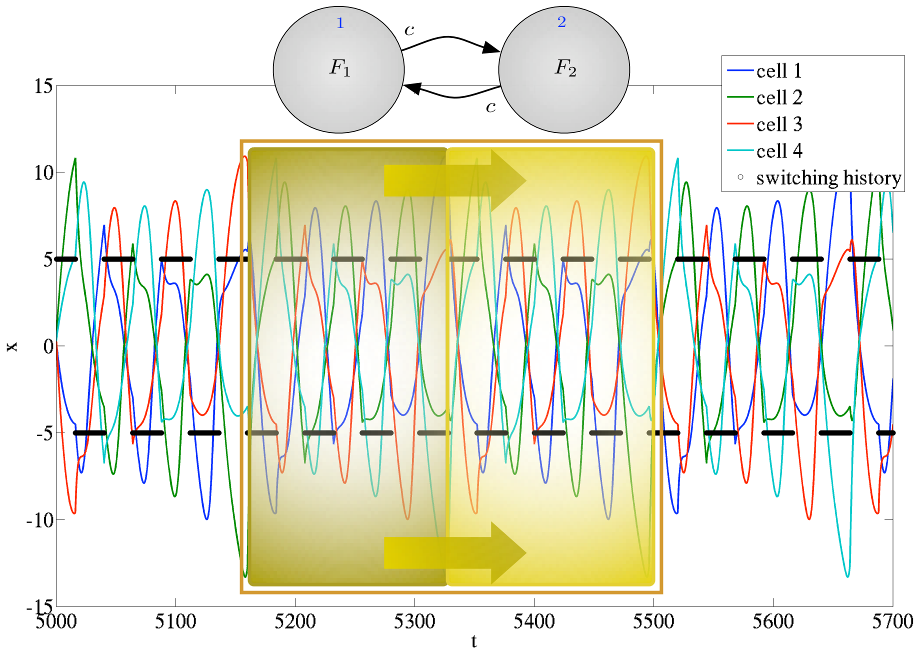


Figure 4.3: Execution with hybrid period $P = (14, 336)$ and non-trivial spatio-temporal symmetries

We characterize spatio-temporal symmetries of an execution $\chi = (\tau, \gamma, x)$ by means of its discrete and continuous data.

4.2.2 Lemma. *An element* $(\Upsilon, P) \in \mathcal{H} \times \Pi_\chi$ *with* $\Upsilon = (\pi, g)$ *and* $P = (N, T)$ *is a spatio-temporal symmetry of* χ *if and only if the relations*

$$\pi^{-1}(\gamma(k-N)) = \gamma(k) \quad \textit{and} \quad g^{-1}_{\gamma(k-N)} x_{k-N}(t-T) = x_k(t) \tag{4.15}$$

hold for all $(k,t) \in \mathcal{T}$.

Proof. Rewrite the equation in (4.14) using Definition 3.4.1 and Eq. (2.14). ■

It should not come as a surprise that the set of hybrid spatio-temporal symmetries exhibits algebraic structure. As in the classical case, there is even more structure to be detected for spatio-temporal symmetries than for purely spatial hybrid symmetries: Spatial and temporal symmetries are knotted in a charateristic manner. For a hybrid dynamical system $\mathscr{H}$ with hybrid symmetries $\mathcal{H}$ and P_χ-periodic execution χ, we set

$$\mathcal{H}_\chi = \{\Upsilon \in \mathcal{H} \mid \mathcal{I}_\chi = \mathcal{I}_{\Upsilon\chi}\} \tag{4.16}$$

to capture the portion of hybrid symmetries that fix the image of χ as a set. Notably, for each $\Upsilon \in \mathcal{H}_\chi$ there is $P \in \Pi_\chi$ such that the pair (Υ, P) is a hybrid spatio-temporal symmetry of χ. Let $\Theta : \mathcal{H}_\chi \to \Pi_\chi$ be the map defined by $\Theta(\Upsilon) = P$.

4.2.3 Proposition. *Let $\mathscr{H}$ be a hybrid dynamical system with hybrid symmetry group $\mathcal{H}$ and P_χ-periodic execution χ. The collection Ξ_χ of spatio-temporal symmetries of χ possesses the following algebraic structure:*

(1) Ξ_χ is a group.

(2) Ξ_χ is a twisted subgroup of $\mathcal{H}_\chi \times \Pi_\chi$ with twist Θ.

(3) The temporal part $\widetilde{\Xi}_\chi \leq \Pi_\chi$ is a twisted subgroup of $\mathbb{Z}/N_\chi\mathbb{Z} \times \mathbb{R}/T_\chi\mathbb{Z}$ with twist $\widetilde{\Theta}$ as defined in Proposition 4.1.6.

Proof. The group structure is easily detected: For two spatio-temporal symmetries $(\Upsilon_i, P_i) \in \Xi_\chi$, $i = 1, 2$, one has

$$(\Upsilon_1\Upsilon_2, P_1 + P_2)\chi(\phi) = \Upsilon_1\Upsilon_2\chi(\phi - P_1 - P_2) = \Upsilon_1\chi(\phi - P_1) = \chi(\phi) \tag{4.17}$$

and the inverse of $(\Upsilon, P) \in \Xi_\chi$ is given by $(\Upsilon, P)^{-1} = (\Upsilon^{-1}, -P)$ which is obviously a spatio-temporal symmetry of χ. Thus, Ξ_χ forms a group. From Eq. (4.17), we see that for $\Upsilon_1, \Upsilon_2 \in \mathcal{H}_\chi$ we have $\Theta(\Upsilon_1\Upsilon_2) = P_1 + P_2 = \Theta(\Upsilon_1) + \Theta(\Upsilon_2)$. Moreover, $\Theta(\Upsilon^{-1}) = -P = -\Theta(\Upsilon)$. Hence, Θ is a group homeomorphism and each element $\xi \in \Xi_\chi$ can be written as $\xi = (\Upsilon, \Theta(\Upsilon))$. Therefore, Ξ_χ is a twisted subgroup of $\mathcal{H}_\chi \times \Pi_\chi$ with twist Θ.

We now turn to the temporal subgroup $\widetilde{\Xi}_\chi \leq \Pi_\chi$ of Ξ_χ. First, we notice that Eq. (4.15) of Lemma 4.2.2 forces

$$|I_{k+N_i}| = |I_k|\,, \qquad i = 1, 2 \tag{4.18}$$

for all $k \in \mathbb{Z}_{\geq 0}$. Being aware of that, we find

$$\Theta_\tau^{(N_1)}(N_2) = \sum_{k=N_1}^{N_1+N_2-1} |I_k| = \sum_{k=0}^{N_2-1} |I_{k+N_1}| \overset{(4.18)}{=} \sum_{k=0}^{N_2-1} |I_k| = \Theta_\tau(N_2) \tag{4.19}$$

and conclude

$$\Theta_\tau(N_1 + N_2) = \Theta_\tau(N_1) + \Theta_\tau^{(N_1)}(N_2) = \Theta_\tau(N_1) + \Theta_\tau(N_2).$$

On the grounds of Proposition 4.1.6, this finally leads to

$$\widetilde{\Theta}_\chi([N_1] + [N_2]) = \widetilde{\Theta}_\chi([N_1]) + \widetilde{\Theta}_\chi([N_2]).$$

Thus, the restriction

$$\widetilde{\Theta}_\chi\Big|_Z : Z \to \mathbb{R}/T_\chi\mathbb{Z} \tag{4.20}$$

with $Z = \mathrm{pr}_{\mathbb{Z}_{N_\chi}}(\widetilde{\Xi}_\chi) \leq \mathbb{Z}/N_\chi\mathbb{Z}$ is a group homomorphism and every $P \in \Pi_\chi$ can be written as

$$P = (N, \widetilde{\Theta}_\chi(N)) \in \mathbb{Z}/N_\chi\mathbb{Z} \times \mathbb{R}/T_\chi\mathbb{Z}. \tag{4.21}$$

Hence, $\widetilde{\Xi}_\chi$ is a twisted subgroup of $\mathbb{Z}/N_\chi\mathbb{Z} \times \mathbb{R}/T_\chi\mathbb{Z}$ with twist $\widetilde{\Theta}_\chi|_Z$. ■

4.2.4 Corollary. *If the discrete period N_χ of χ is prime, $\widetilde{\Xi}_\chi$ is either trivial or isomorphic to $\mathbb{Z}_{N_\chi}$.*

Proof. The assumption that N_χ is prime forces $N = 1$ or $N = N_\chi$, since for $N \in \mathbb{Z}/N_\chi\mathbb{Z}$, one has $N\mathrm{ord}(N) = N_\chi$ and thus N divides N_χ. Hence, $[N]_{N_\chi} = 1$ or $[N]_{N_\chi} = 0$ and Proposition 4.2.3 entails $\widetilde{\Xi}_\chi \cong \mathbb{Z}_{N_\chi}$ or $\widetilde{\Xi}_\chi = 1$, respectively. ■

Having clarified the basic algebraic circumstances under which our current considerations take place, we now aim to establish a connection between the hybrid spatio-temporal symmetries of an execution χ of a hybrid dynamical system $\mathscr{H}$ and the hybrid symmetries of the induced hybrid system $\mathscr{H}_\chi$.

4.2.5 Lemma. *For a hybrid dynamical system $\mathscr{H}$ with hybrid symmetries $\mathcal{H}$, let χ be a periodic execution with minimal period P_χ and spatio-temporal symmetries Ξ_χ. Then the map*

$$\widehat{r}_\chi : \widehat{\Xi}_\chi \to \mathrm{Aut}\,(\mathscr{T}_\chi)\,, \quad \pi \mapsto \pi|_{\Lambda_\chi}\,, \tag{4.22}$$

is a well-defined group homomorphism, where $\widehat{\Xi}_\chi \leq \widehat{\mathcal{H}}$ is the obvious projection.

Proof. For $\pi \in \widehat{\Xi}_\chi$, there is $N \in \mathbb{N}$ such that the identity $\pi^{-1}(\gamma(k)) = \gamma(k+N)$ holds for all $k \in \mathbb{Z}$ thus showing that Λ_χ is $\widehat{\Xi}_\chi$-invariant. Hence π restricts to a bijection $\pi|_{\Lambda_\chi} : \Lambda_\chi \to \Lambda_\chi$. For $e_k = (\gamma(k), \gamma(k+1)) \in \mathcal{E}_\chi$, one has

$$\begin{aligned}\pi|_{\Lambda_\chi}^{-1}(e_k) &= \left(\pi|_{\Lambda_\chi}^{-1}(\gamma(k)), \pi|_{\Lambda_\chi}^{-1}(\gamma(k+1))\right) \\ &= (\gamma(k+N), \gamma(k+N+1)) \\ &= e_{k+N} \in \mathcal{E}_\chi\end{aligned}$$

and so $\pi|_{\Lambda_\chi}$ is adjacency-preserving. Therefore, $\pi|_{\Lambda_\chi} \in \mathrm{Aut}\,(\mathscr{T}_\chi)$ proving that r_χ is well-defined. Clearly, $(\pi_1 \circ \pi_2)\,|_{\Lambda_\chi} = \pi_1|_{\Lambda_\chi} \circ \pi_2|_{\Lambda_\chi}$ as well as $\pi|_{\Lambda_\chi}^{-1} = (\pi^{-1})\,|_{\Lambda_\chi}$ for all $\pi, \pi_1, \pi_2 \in \widehat{\Xi}_\chi$ characterizing $\widehat{r}_\chi$ as a group homomorphism. ∎

Using this statement, we can relate the spatio-temporal symmetry properties of an execution χ to the hybrid symmetries of the induced hybrid system $\mathscr{H}_\chi$.

4.2.6 Proposition. *Let $\chi = (\tau, \gamma, x)$ be a periodic execution with minimal period $P_\chi = (N_\chi, T_\chi)$ and $(\Upsilon, P) \in \Xi_\chi$ a spatio-temporal symmetry of χ where $\Upsilon = (\pi, g)$ and $P = (N, T)$. Then Υ induces a hybrid symmetry $\widetilde{\Upsilon} = (\widetilde{\pi}, \widetilde{g})$ of the induced system $\mathscr{H}_\chi$ with $\widetilde{\pi} = \pi|_{\Lambda_\chi}$.*

Proof. We define $\widetilde{\Upsilon} = (\widetilde{\pi}, \widetilde{g})$ by $\widetilde{\pi} = \widehat{r}_\chi(\pi)$ and $\widetilde{g} = \widehat{r}_\chi(g)$ where the map $\widehat{r}_\chi$ is the projection

$$\widehat{\Xi}_\chi \to \prod_{\lambda \in \Lambda_\chi} G_\lambda, \quad g \mapsto \widetilde{g} \quad \text{with} \quad \widetilde{g}_\lambda = g_\lambda.$$

We verify that $\widetilde{\Upsilon}$ is a hybrid symmetry of $\mathscr{H}_\chi$. First of all, for $\lambda \in \Lambda_\chi$ we see that

$$\Theta_\chi\left(\widetilde{\pi}^{-1}(\lambda)\right) = \Theta(\pi^{-1}(\lambda)) = \Theta(\lambda) = \Theta_\chi(\lambda)$$

and

$$\begin{aligned}\mathcal{F}_\chi\left(\widetilde{\pi}^{-1}(\lambda)\right) \circ \widetilde{g}_\lambda^{-1} &= \mathcal{F}\left(\pi^{-1}(\lambda)\right) \circ g_\lambda^{-1} \\ &= g_\lambda^{-1} \circ \mathcal{F}(\lambda) \\ &= \widetilde{g}_\lambda^{-1} \circ \mathcal{F}_\chi(\lambda).\end{aligned}$$

Moreover, we have

$$\widetilde{g}_{\widetilde{\pi}^{-1}(\lambda)} = g_\pi^{-1}(\lambda) = g_\lambda = \widetilde{g}_\lambda$$

for all $\lambda \in \Lambda_\chi$. Hence, $\widetilde{\pi}^* \widetilde{g} = \widetilde{g}$ for all $\nu \in \widehat{\mathcal{H}}_\chi$ and thus $\widetilde{\Upsilon}$ is a $\mathscr{T}_\chi$-symmetry. Lastly, we find

$$\begin{aligned}
\widetilde{\Upsilon}\mathcal{R}_\chi(e_k) &= \widetilde{g}_{\widetilde{\pi}^{-1}(\mathfrak{s}e_k)}\mathcal{R}_\chi\left(\widetilde{\pi}^{-1}(e_k)\right) \\
&= g_{\pi^{-1}(\mathfrak{s}e_k)}\mathcal{R}\left(\pi^{-1}(e_k)\right) \\
&= \mathcal{R}(e_k) \\
&= \mathcal{R}_\chi(e_k)
\end{aligned}$$

for all $e_k \in \mathcal{E}_\chi$ finally proving that $\widetilde{\Upsilon}$ is a hybrid symmetry of $\mathscr{H}_\chi$. ■

This provides us with the following algebraic connection between hybrid spatio-temporal symmetries and hybrid symmetries of the induced hybrid system.

4.2.7 Corollary. *For a periodic execution χ with hybrid spatio-temporal symmetries Ξ_χ the map*

$$r_\chi = (\widehat{r}_\chi, \widehat{r}_\chi) : \Xi_\chi^{\mathcal{H}} \to \mathcal{H}_\chi, \quad \Upsilon = (\pi, g) \mapsto \widetilde{\Upsilon} = (\widehat{r}_\chi(\pi), \widehat{r}_\chi(g)) \tag{4.23}$$

is a well-defined group homomorphism. Here, $\Xi_\chi^{\mathcal{H}}$ is the projection of Ξ_χ onto $\mathcal{H}$ and $\mathcal{H}_\chi$ is the hybrid symmetry group of $\mathscr{H}_\chi$.

In order to explore the nature of hybrid spatio-temporal symmetries further, we lift such symmetries to the hybrid trace $\mathscr{H}_\chi^*$.

4.2.8 Proposition. *Let $\chi = (\tau, \gamma, x)$ be a periodic execution with minimal period $P_\chi = (N_\chi, T_\chi)$ and $(\Upsilon, P) \in \Xi_\chi$ a spatio-temporal symmetry of χ where $\Upsilon = (\pi, g)$ and $P = (N, T)$. Then the hybrid symmetry Υ induces a hybrid symmetry $\widetilde{\Upsilon} = (\widetilde{\pi}, \widetilde{g})$ of the hybrid trace $\mathscr{T}_\chi^*$ of χ. Moreover, $\operatorname{ord}(\widetilde{\Upsilon}) = \operatorname{ord}(\widetilde{\pi}) = \operatorname{ord}(\Upsilon) = \kappa$ with $\kappa = \frac{N_\chi}{N} = \frac{T_\chi}{T}$.*

Proof. We define $\widetilde{\pi}$ by

$$\Lambda_\chi^* \to \Lambda_\chi^*, \quad \widetilde{\pi}^{-1}\left([k]_{N_\chi}, \gamma(k)\right) = \left([k+N]_{N_\chi}, \pi^{-1}(\gamma(k))\right). \tag{4.24}$$

Via the identity $\pi^{-1}(\gamma(k-N)) = \gamma(k)$ we get $\pi^{-1}(\gamma(k)) = \gamma(k+N)$ for all k and by cyclicity of $\mathscr{T}_\chi^*$, $\widetilde{\pi}$ turns out to be an automorphism of $\mathscr{T}_\chi^*$. For $\kappa_N = \frac{N_\chi}{N}$, we compute

$$\begin{aligned}
\widetilde{\pi}^{-\kappa_N}\left([k]_{N_\chi}, \gamma(k)\right) &\overset{(4.24)}{=} \left([k+\kappa_N N]_{N_\chi}, \pi^{-\kappa_N}(\gamma(k))\right) \\
&= \left([k+N_\chi]_{N_\chi}, \gamma(k+\kappa_N N)\right) \\
&= \left([k]_{N_\chi}, \gamma(k)\right),
\end{aligned} \tag{4.25}$$

using N_χ-periodicity in the last step. Thus, $\operatorname{ord}(\widetilde{\pi}) \leq \kappa_N$. Since κ_N is the minimal positive non-zero integer satisfying $[k+\kappa_N N]_{N_\chi} = [k]_{N_\chi}$, we see that $\operatorname{ord}(\widetilde{\pi}) \geq \kappa_N$, finally yielding $\operatorname{ord}(\widetilde{\pi}) = \kappa_N$. For the continuous part, we set

$$\widetilde{g} \in \prod_{k=0}^{N_\chi - 1} G_{\lambda_k} \quad \text{with} \quad \widetilde{g}_{\lambda_k} = g_{\gamma(k)}, \quad \lambda_k = \Big([k]_{N_\chi}, \gamma(k) \Big) \in \Lambda_\chi^*, \tag{4.26}$$

for all $k \in \{0, \ldots, N_\chi - 1\}$. We show that $\widetilde{\Upsilon} = (\widetilde{\pi}, \widetilde{g})$ is a hybrid symmetry of $\mathscr{H}_\chi^*$: We have

$$\begin{aligned}
\Theta^*\left(\widetilde{\pi}^{-1}(\lambda_k)\right) &= \Theta^*\Big([k+N]_{N_\chi}, \pi^{-1}(\gamma(k)) \Big) \\
&= \Theta\left(\pi^{-1}(\gamma(k))\right) \\
&= \Theta\left(\gamma(k)\right) \\
&= \Theta^*(\lambda_k)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}^*\left(\widetilde{\pi}^{-1}(\lambda_k)\right) \circ \widetilde{g}_{\lambda_k}^{-1} &= \mathcal{F}\left(\pi^{-1}(\gamma(k))\right) \circ g_{\gamma(k)}^{-1} \\
&= g_{\gamma(k)}^{-1} \circ \mathcal{F}(\gamma(k)) \\
&= \widetilde{g}_{\lambda_k}^{-1} \circ \mathcal{F}^*\left(\lambda_k\right)
\end{aligned}$$

for all $k \in \{0, \ldots, N_\chi - 1\}$. Moreover,

$$\widetilde{g}_{\widetilde{\pi}^{-1}(\lambda_k)} = g_{\pi^{-1}(\gamma(k))} = g_{\gamma(k)} = \widetilde{g}_{\lambda_k},$$

which indeed holds for all $\widetilde{\pi} \in \widehat{\mathfrak{S}}_\chi^*$. Thus, $\widetilde{\Upsilon}$ is a $\mathscr{T}_\chi^*$-symmetry of the according $\mathscr{T}_\chi^*$-system. Finally,

$$\begin{aligned}
\widetilde{\Upsilon}\mathcal{R}^*(\lambda_k, \lambda_{k+1}) &= \widetilde{g}_{\widetilde{\pi}^{-1}(\lambda_k)}^{-1} \mathcal{R}^*(\lambda_k, \lambda_{k+1}) \\
&= g_{\pi^{-1}(\gamma(k))}^{-1} \mathcal{R}(\gamma(k), \gamma(k+1)) \\
&= \mathcal{R}\left(\gamma(k), \gamma(k+1)\right) \\
&= \mathcal{R}^*\left(\lambda_k, \lambda_{k+1}\right)
\end{aligned}$$

verifying that $\widetilde{\Upsilon}$ is a hybrid symmetry of $\mathscr{H}_\chi^*$.

We now consider the map $\widetilde{\Upsilon} = (\widetilde{\pi}, g) : \mathbb{Z}_{N_\chi} \times D_\chi \to \mathbb{Z}_{N_\chi} \times D_\chi$ which is induced by the hybrid symmetry Υ and given by

$$\widetilde{\Upsilon}\Big([k]_{N_\chi}, \gamma(k), x_{\gamma(k)}(t) \Big) = \Big(\widetilde{\pi}^{-1}\Big([k]_{N_\chi}, \gamma(k) \Big), g_{\pi^{-1}(\gamma(k))}^{-1} x_{\gamma(k)}(t) \Big). \tag{4.27}$$

Let $\kappa_T = \frac{T_\chi}{T}$. From $N_\chi = \kappa_N N$ we obtain

$$T_\chi = \Theta_\tau(N_\chi) = \kappa_N \Theta_\tau(N) = \kappa_N T,$$

and, consequently, $\kappa_T = \kappa_N = \kappa$. We carry on computing

$$\begin{aligned}
\widetilde{\Upsilon}^\kappa \left([k]_{N_\chi}, \gamma(k), x_{\gamma(k)}(t)\right) &= (\widetilde{\pi}^{\kappa_N}, g^{\kappa_T}) \left([k]_{N_\chi}, \gamma(k), x_{\gamma(k)}(t)\right) \\
&\overset{(4.27)}{=} \left(\widetilde{\pi}^{-\kappa_N}\left([k]_{N_\chi}, \gamma(k)\right), g^{-\kappa_T}_{\pi^{-\kappa_N}(\gamma(k))} x_{\gamma(k)}(t)\right) \\
&\overset{(4.25),\ (4.15)}{=} \left([k]_{N_\chi}, \gamma(k), x_{\gamma(k-\kappa_N N)}(t - \kappa_T T)\right) \\
&= \left([k]_{N_\chi}, \gamma(k), x_{\gamma(k)}(t)\right).
\end{aligned}$$

From $\mathrm{ord}(\widetilde{\pi}) = \kappa$, we can conclude that $\mathrm{ord}(\widetilde{\Upsilon}) = \kappa$ and similarly $\mathrm{ord}(\Upsilon) = \kappa$. Note – however – that $\mathrm{ord}(g) \leq \kappa_T = \kappa$ and not necessarily $\mathrm{ord}(g) = \kappa$. ∎

Similar to Corollary 4.2.7, we obtain the following statement.

4.2.9 Corollary. *For a periodic execution χ with hybrid spatio-temporal symmetries Ξ_χ, the map*

$$r^*_\chi = \left(\widehat{r}^*_\chi, \widehat{r}^*_\chi\right) : \Xi^{\mathcal{H}}_\chi \to \mathcal{H}^*_\chi, \quad \Upsilon = (\pi, g) \mapsto \widetilde{\Upsilon} = \left(\widehat{r}^*_\chi(\pi), \widehat{r}^*_\chi(g)\right) \tag{4.28}$$

is a well-defined group homomorphism.

This concludes the structural study of hybrid spatio-temporal symmetries and we pass on to question how to utilize this structure for the dynamical analysis.

4.3 Return Maps and Hybrid Symmetries

It is the task of this section to turn away from the rather narrow view of the preceding considerations and to expand the horizon towards a more global understanding of hybrid spatio-temporal symmetry phenomena: Accordingly, we step from *individual periodic executions* together with their hybrid spatio-temporal symmetry groups on to hybrid dynamical systems equipped with a periodic switching signal giving rise to spatio-temporal symmetries of the *system*. Thus, we pass on to a more broadly based treatment of spatio-temporal symmetries in which the foregoing perception will finally turn up again. At this point, it should be stressed that a periodic execution clearly holds a periodic switching signal and, hence, naturally blends in with the upcoming findings.

We consider a hybrid dynamical system $\mathscr{H}$ with hybrid symmetries $\mathcal{H}$. Let $\sigma = (\tau, \gamma)$ be a switching signal for $\mathscr{H}$ of period N_σ, i.e. $\gamma(k+N_\sigma) = \gamma(k)$ and $I_{k+N_\sigma} = I_k + \Theta_\tau(N_\sigma)$ for all $k \in \mathbb{Z}$. Then σ gives rise to the switched system $\mathscr{H}_\sigma$ modelled on the induced transition graph $\mathscr{T}_\sigma = (\Lambda_\sigma, \mathcal{E}_\sigma)$. The hybrid phase space D_σ of $\mathscr{H}_\sigma$ is given by $D_\sigma = \bigcup_{\lambda \in \Lambda_\sigma} \mathcal{D}(\lambda)$ where $\Lambda_\sigma = \mathrm{im}(\gamma) \subset \Lambda$.

4.3.1 Definition. Let σ be an N_σ-periodic switching signal. A hybrid spatio-temporal symmetry of $\mathscr{H}_\sigma$ is a pair $(\Upsilon, N) \in \mathcal{H} \times \mathbb{Z}/N_\sigma\mathbb{Z}$ with $\Upsilon = (\pi, g)$ satisfying

$$(\Upsilon, N)\, \mathcal{F}(\gamma(k)) = g_{\gamma(k)} \circ \mathcal{F}\left(\gamma(k+N)\right) \circ g^{-1}_{\gamma(k+N)} = \mathcal{F}(\gamma(k)) \tag{4.29}$$

and

$$\pi^{-1}\left(\gamma(k-N)\right) = \gamma(k) \tag{4.30}$$

for all $k \in \mathbb{Z}$. ♢

Note that on account of the constancy condition $\pi^* g = g$ we have

$$g_{\gamma(k)} = g_{\pi^{-1}(\gamma(k))} = g_{\gamma(k+N)} \tag{4.31}$$

for all $k \in \mathbb{Z}$ ensuring that (4.29) is well-defined.

4.3.2 Lemma. *Let σ be a N_σ-periodic switching signal and (Υ, N) a hybrid spatio-temporal symmetry of the induced hybrid system $\mathscr{H}_\sigma$. Then $g_{\gamma(\cdot)}$ is constant on $\mathbb{Z}$ if and only if $g_{\gamma(0)} = g_{\gamma(1)} = \cdots = g_{\gamma(N-1)}$.*

Proof. Successive application of (4.31) leads to

$$g_{\gamma(k)} = g_{\gamma(k+nN)} \tag{4.32}$$

for all $k, n \in \mathbb{Z}$. Especially, for $n = 1$, we obtain $g_{\gamma(k+N)} = g_{\gamma(k)}$. Since every $k \in \mathbb{Z}$ can be written in the form

$$k = k_0 + k_1 N + k_2 N_\sigma$$

with $k_0 \in \mathcal{I}_0 = \{0, \pm 1, \dots, \pm(N-1)\}$, $k_1 \in \mathcal{I}_1 = \left\{0, \pm 1, \dots, \pm \frac{N_\sigma}{N}\right\}$ and $k_2 \in \mathbb{Z}$, we find

$$g_{\gamma(k)} = g_{\gamma(k_0 + k_1 N + k_2 N_\sigma)} = g_{\gamma(k_0 + k_1 N)} = g_{\gamma(k_0)}$$

already proving the equivalence of statements. ∎

The group of hybrid spatio-temporal symmetries of $\mathscr{H}_\sigma$ is denoted by Ξ_σ. Let $\phi^\lambda : \mathbb{T} \times \mathcal{D}(\lambda) \to \mathcal{D}(\lambda)$ denote the flow of the vector field $\mathcal{F}(\lambda)$ corresponding to the discrete state $\lambda \in \Lambda$. For a switching signal $\sigma = (\tau, \gamma)$ of $\mathscr{H}$, note that the induced hybrid dynamical system $\mathscr{H}_\sigma$ is deterministic and non-blocking. Therefore, for every $p \in D_\sigma$, there exists a unique infinite execution $\chi = (\sigma, x)$ with $\chi(0,0) = p$. Moreover, for every $t \in \mathbb{R}$ there is a unique index $k \in \mathbb{Z}$ with $t \in I_k^{)}$, i.e. $(k,t) \in \mathcal{T}$. We define the *hybrid flow* of $\mathscr{H}_\sigma$ by

$$\Phi^\sigma : \mathbb{T} \times D_\sigma \to D_\sigma, \quad \Phi(t,p) = \Phi_t(p) = (\gamma(k), x_k(t)). \tag{4.33}$$

Let $\widehat{\phi}^\lambda$ and $\widehat{\Phi}^\sigma$ denote just the continuous part of ϕ^λ and Φ^σ, respectively. Due to the special structure of hybrid systems, the hybrid evolution operator Φ_I decomposes as follows for some interval $I \subset \mathbb{R}$:

4.3.3 Proposition. *Let $\sigma = (\tau, \gamma)$ be a switching signal for $\mathscr{H}$. Then for an interval $I = [a,b] \subset \mathbb{R}$, one has*

$$\Phi_I^\sigma = \phi_{b-\tau_k'}^{\gamma(k)} \circ \phi_{c_{k-1}}^{\gamma(k-1)} \circ \cdots \circ \phi_{c_{l+1}}^{\gamma(l+1)} \circ \phi_{\tau_l - a}^{\gamma(l)} \tag{4.34}$$

with $k, l \in \mathbb{Z}$ being defined by $b \in I_k^{)}$ and $a \in I_l^{)}$. If σ is N_σ-periodic and $(\Upsilon, N) \in \Xi_\sigma$ is a hybrid spatio-temporal symmetry of the induced hybrid system $\mathscr{H}_\sigma$ with $\Upsilon = (\pi, g)$ such that $g_{\gamma(k)} = g_{\gamma(k+1)}$ for all $k \in \mathbb{Z}$, then with $T = \Theta_\tau(N)$

$$\Phi^\sigma_{[(n-1)T, nT]} = \Upsilon^{-k} \circ \Phi^\sigma_{[(n+k-1)T, (n+k)T]} \circ \Upsilon^k \tag{4.35}$$

holds for all $k, n \in \mathbb{Z}$.

Proof. For $a, b \in \mathbb{R}$, there exists a uniquely determined pair $(k,l) \in \mathbb{Z} \times \mathbb{Z}$ of indices such that $a \in I_l^{)} = [\tau_l, \tau_l')$ and $b \in I_k^{)} = [\tau_k, \tau_k')$. Looking at (4.33), we immediately see that Φ_I may be written as the composition

$$\Phi^\sigma_{[a,b]} = \phi_{b-\tau_k'}^{\gamma(k)} \circ \phi_{c_{k-1}}^{\gamma(k-1)} \circ \cdots \circ \phi_{c_{l+1}}^{\gamma(l+1)} \circ \phi_{\tau_l - a}^{\gamma(l)}$$

with $c_j = |I_j| = \tau'_j - \tau_j$. First, observe that for a hybrid symmetry $\Upsilon = (\pi, g)$ (4.29) carries over to the hybrid flow in form of

$$\phi_{|I|}^{\gamma(k)} = \Upsilon^{-1} \circ \phi_{|I+\Theta_\tau(N)|}^{\gamma(k+N)} \circ \Upsilon, \tag{4.36}$$

where we use the equivalent relations $\pi^{-1}(\gamma(k)) = \gamma(k+N)$ and $\pi(\gamma(k+N)) = \gamma(k)$ and recall that hybrid symmetries act via $\Upsilon(\lambda, x) = \left(\pi^{-1}(\lambda), g_{\pi^{-1}(\lambda)}^{-1} x\right)$. Note that on the continuous level Eq. (4.36) takes the form

$$\widehat{\phi}_{|I|}^{\gamma(k)} = g_{\gamma(k)} \circ \widehat{\phi}_{|I+\Theta_\tau(N)|}^{\gamma(k+N)} \circ g_{\gamma(k+N)}^{-1}. \tag{4.37}$$

Set $T = \Theta_\tau(N)$. Applying Eq. (4.36), we obtain

$$\begin{aligned}
\widehat{\Phi}_{[(n-1)T,nT]}^{\sigma} &= \widehat{\phi}_{|I_{nN-1}|}^{\gamma(nN-1)} \circ \widehat{\phi}_{|I_{nN-2}|}^{\gamma(nN-2)} \circ \cdots \circ \widehat{\phi}_{|I_{(n-1)N+1}|}^{\gamma((n-1)N+1)} \circ \widehat{\phi}_{|I_{(n-1)N}|}^{\gamma((n-1)N)} \\
&= \left(g_{\gamma(nN-1)} \circ \widehat{\phi}_{|I_{nN-1}+T|}^{\gamma((n+1)N-1)} \circ g_{\gamma((n+1)N-1)}^{-1}\right) \\
&\quad \circ \left(g_{\gamma(nN-2)} \circ \widehat{\phi}_{|I_{nN-2}+T|}^{\gamma((n+1)N-2)} \circ g_{\gamma((n+1)N-2)}^{-1}\right) \circ \ldots \\
&\quad \circ \left(g_{\gamma((n-1)N+1)} \circ \widehat{\phi}_{|I_{(n-1)N+1}+T|}^{\gamma(nN+1)} \circ g_{\gamma(nN+1)}^{-1}\right) \\
&\quad \circ \left(g_{\gamma((n-1)N)} \circ \widehat{\phi}_{|I_{(n-1)N}+T|}^{\gamma(nN)} \circ g_{\gamma(nN)}^{-1}\right) \\
&= g_{\gamma(nN-1)} \circ \widehat{\phi}_{|I_{(n+1)N-1}|}^{\gamma((n+1)N-1)} \circ \widehat{\phi}_{|I_{(n+1)N-2}|}^{\gamma((n+1)N-2)} \circ \ldots \\
&\quad \circ \widehat{\phi}_{|I_{nN+1}|}^{\gamma(nN+1)} \circ \widehat{\phi}_{|I_{nN}|}^{\gamma(nN)} \circ g_{\gamma(nN)}^{-1}
\end{aligned}$$

on the continuous side, where we use $|I_k + T| = |I_{k+N}|$ and the constancy condition $g_{\gamma(k)} = g_{\gamma(k')}$ for all $k, k' \in \mathbb{Z}$. In the hybrid notation along the lines of Eq. (4.36), the above identity reads

$$\Phi_{[(n-1)T,nT]}^{\sigma} = \Upsilon^{-1} \circ \Phi_{[nT,(n+1)T]}^{\sigma} \circ \Upsilon. \tag{4.38}$$

Hence, via repeated application of this equation we end up with

$$\Phi_{[(n-1)T,nT]}^{\sigma} = \Upsilon^{-k} \circ \Phi_{[(n+k-1)T,(n+k)T]}^{\sigma} \circ \Upsilon^{k} \tag{4.39}$$

as stated above, thus completing the proof. ■

The preceding proposition yields the basis for a structural decomposition of the hybrid time-T_σ map on the grounds of hybrid spatio-temporal symmetries. This statement is an adaption for the hybrid case of Jeroen Lamb's work on smooth non-autonomous dynamical systems exhibiting periodicity in time (see [Lam98]).

4.3.4 Theorem. *Let $\sigma = (\tau, \gamma)$ be a N_σ-periodic switching signal and $\mathscr{H}_\sigma$ the induced switched system. Let $(\Upsilon, N) \in \Xi_\sigma$ be a hybrid spatio-temporal symmetry of $\mathscr{H}_\sigma$ with $\Upsilon = (\pi, g)$ such that $g_{\gamma(\cdot)}$ is constant on $\mathbb{Z}$. Then, the hybrid time-$\Theta(N_\sigma)$ map $\Phi_{\Theta_\tau(N_\sigma)} = \Phi_{[0,\Theta_\tau(N_\sigma)]}$ admits the following decomposition:*

$$\Phi^\sigma_{\Theta_\tau(N_\sigma)} = \Upsilon^\kappa \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_{\Theta_\tau(N)}\right)^\kappa \tag{4.40}$$

with $\kappa = \frac{N_\sigma}{N}$.

Proof. For $T_\sigma = \Theta_\tau(N_\sigma)$, we have

$$\Phi^\sigma_{T_\sigma} = \Phi^\sigma_{[(\kappa-1)T,T_\sigma]} \circ \Phi^\sigma_{[(\kappa-2)T,(\kappa-1)T]} \circ \cdots \circ \Phi^\sigma_{[T,2T]} \circ \Phi^\sigma_{[0,T]} \tag{4.41}$$

and since

$$\Phi^\sigma_{[(n-1)T,nT]} = \Upsilon^{-k} \circ \Phi^\sigma_{[(n+k-1)T,(n+k)T]} \circ \Upsilon^k \tag{4.42}$$

holds due to Proposition 4.3.3, we see

$$\begin{aligned}
\Phi^\sigma_{T_\sigma} &= \left(\Upsilon^{\kappa-1} \circ \Phi^\sigma_T \circ \Upsilon^{-\kappa+1}\right) \circ \left(\Upsilon^{\kappa-2} \circ \Phi^\sigma_T \circ \Upsilon^{-\kappa+2}\right) \circ \ldots \\
&\quad \circ \left(\Upsilon \circ \Phi^\sigma_T \circ \Upsilon^{-1}\right) \circ \Phi^\sigma_T \\
&= \Upsilon^{\kappa-1} \circ \Phi^\sigma_T \circ \Upsilon^{-1} \circ \Phi^\sigma_T \circ \cdots \circ \Upsilon^{-1} \circ \Phi^\sigma_T \circ \Upsilon^{-1} \circ \Phi^\sigma_T
\end{aligned}$$

for the pairs $(n,k) \in \mathbb{Z} \times \mathbb{Z}$ with $n = 1, \ldots, \kappa$ and $n - k = 1$ or, equivalently,

$$\begin{aligned}
\Phi^\sigma_{T_\sigma} &= \Upsilon^\kappa \circ \underbrace{\Upsilon^{-1} \circ \Phi^\sigma_T \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right) \circ \cdots \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right) \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right)}_{\kappa \text{ times}} \\
&= \Upsilon^\kappa \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right)^\kappa,
\end{aligned}$$

yielding the desired statement. ∎

From a structural point of view, Definition 4.2.1 and 4.3.1 appear to introduce quite similar concepts. This is indeed so and we will now render this similarity more precisley and bring together the two manifestations of spatio-temporal symmetries.

4.3.5 Proposition. *Let χ be an execution of $\mathscr{H}$ with respect to an N_σ-periodic switching signal $\sigma = (\tau, \gamma)$. Let $(\Upsilon, N) \in \Xi_\sigma$ be a hybrid spatio-temporal symmetry of $\mathscr{H}_\sigma$ with $\Upsilon = (\pi, g)$ such that $g_{\gamma(\cdot)}$ is constant on $\mathbb{Z}$. For $F = F(\sigma, \Upsilon, N) = \Upsilon^{-1} \circ \Phi^\sigma_{\Theta_\tau(N)}$, one has*

$$F \circ \Phi^\sigma_{T_\sigma} = \Phi^\sigma_{T_\sigma} \circ F. \tag{4.43}$$

Further, let be $p = \chi(k,t)$ for some $(k,t) \in \mathcal{T}$. Then p is an n-periodic point of F if and only if $(\Upsilon, N)^n$ is a hybrid spatio-temporal symmetry of χ.

Proof. Using Theorem 4.3.4, with $T = \Theta_\tau(N)$ we obtain

$$F \circ \Phi^\sigma_{T_\sigma} = \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right) \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right)^\kappa = \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right)^\kappa \circ \left(\Upsilon^{-1} \circ \Phi^\sigma_T\right) = \Phi^\sigma_{T_\sigma} \circ F.$$

For $(k,t) \in \mathcal{T}$ and $p = \chi(k,t)$, we compute

$$\begin{aligned}
F^n(p) &= F^{n-1}\left(\Upsilon^{-1}\Phi^\sigma_T(p)\right) \\
&= F^{n-1}\left(\Upsilon^{-1}\chi(k+N, t+T)\right) \\
&= F^{n-1}\left(\pi(\gamma(k+N)), g_{\gamma(k+N)}x_{\gamma(k+N)}(t+T)\right) \\
&\overset{(4.30)}{=} F^{n-1}\left(\gamma(k), g_{\gamma(k+N)}x_{\gamma(k+N)}(t+T)\right) \\
&= F^{n-2}\left(\left(\Upsilon^{-1} \circ \Phi^\sigma_T\right)\left(\gamma(k), g_{\gamma(k+N)}x_{\gamma(k+N)}(t+T)\right)\right) \\
&= F^{n-2}\left(\Upsilon^{-1}\left(\gamma(k+N), g_{\gamma(k+2N)}x_{\gamma(k+2N)}(t+2T)\right)\right) \\
&= F^{n-2}\left(\gamma(k), g^2_{\gamma(k+2N)}x_{\gamma(k+2N)}(t+2T)\right) \\
&= \dots \\
&= \left(\gamma(k), g^n_{\gamma(k+N)}x_{\gamma(k+nN)}(t+nT)\right),
\end{aligned}$$

using the relation $\Phi^\sigma_{[mT,(m+1)T]} = g^{-m}_{\gamma(\cdot)} \circ \Phi^\sigma_{[0,T]} \circ g^m_{\gamma(\cdot)}$ (cp. Proposition 4.3.3) in the sixth step. Thus, if $F^n(p) = p$, then $(\Upsilon^n, nN) = (\Upsilon, N)^n$ is a spatio-temporal symmetry of χ (and vice versa) by means of Lemma 4.2.2. ∎

Chapter 5

Symmetry Switching

As already discussed in earlier parts of this thesis, hybrid dynamical systems compared to non-autonomous dynamical systems have a tendency to be *non-deterministic* in the sense that the sequence of vector fields may vary due to different underlying discrete state maps. Treating hybrid dynamical systems theoretically, it is – from a structural point of view – unfavorable to extract the switching laws from executions since there may be many executions guided by the same switching sequence. Thus in order to obtain a broader view on the structural nature of a hybrid system's dynamics, we shift our point of view as follows: Instead of examining a hybrid automaton we consider an according *switched system* which may be understood as a dynamical $\mathscr{T}$-system equipped with a collection of *switching signals* each of which induces a hybrid automaton whose dynamics can be studied as in the preceding chapters. The essential advantage of this way of perception lies in the possibility to treat a hybrid dyamical system as a system that allows for different *hybrid* modes of dynamics: Certainly, in the first instance, hybridity is locally caused by the abrupt change of vector fields; nonetheless, hybridity also appears on a global scale in form of the various switching signals, i. e. the *divers coexistant* manners of varying the vector fields. Approaching this point philosophically, one might say that a hybrid dynamical system is characterized by *different specifications of time* each of which is realized by an according switching signal.

The core subject of this chapter is the analysis of a specific type of switching referred to as *symmetry switching* or *orbital switching* which is characterized by the fact that it is generated by internal structural information of the overall hybrid system itself, more specifically by hybrid symmetries. Signals of this kind may thus be understood as a manifestation of the system's genetical inside and may thence be classified to be of *self-organized switching* type. According to the general considerations above, such signals give rise to a temporal structure which could be referred to as *eigentime* in a broader not strictly mathematical meaning.

Stability is a highly important concept in the field of dynamical systems which – with a view to applications – is closely connected to questions of safety and relieability. In particular consideration of hybrid systems, whose dynamical behaviour strongly depends on the current switching sequence that may be generated by the system itself under state-dependent switching or chosen in advance for time-dependent switching, stability gains an even more central status since in principle it can be accessed more directly than in the case of a classical dynamical system. While in the latter case stability is an inherent property of the system, for hybrid systems the situation is considerably different since the (inherent) stability properties of the involved subsystems do not determine the hybrid system's stability properties on their own but deeply interact with the switching law at work. Consequently, already the adequate choice of switching signals may lead to manifestations of stability. Stability of hybrid systems is the subject of numerous publications; among the leading survey articles on this matter, there are [LM99], [DK01], [SWM$^+$07] and [LA09]. Most of the contributions to the stability analysis of hybrid systems can be traced back to two principal methods involving multiple Lyapunov functions on the one hand, see e. g. [CGA05], and the notion of dwell time discussed in [HM99], for instance. Besides, there are a few different approaches as presented in [YCJ09] where also unstables modes are considered and [AL01] and [ML06] where Lie-algebraic stability conditions are developed. In [MRA$^+$07], connections between the chosen switching strategy of a hybrid system and its stability are investigated for a class of identical chemical reactor networks. As in [MRB$^+$07], temporally induced symmetry properties arise and are exploited for the numerical analysis.

First of all, Chapter 5.1 centers on *switched* systems against the background of hybrid dynamical systems as treated before emphasizing the role of switching signals. In Chapter 5.2, we briefly review the concept of hybrid symmetries as introduced in Chapter 3 for the case of switched systems where we incorporate the slowness of signals and address the question concerning symmetries of signal-induced systems. Subsequently, we focus on *orbital* switching which specifies a subclass of signals that are generated by hybrid symmetries. In this context, we discuss the orbitally induced switched systems and the decomposition of return maps, always taking *slowness* – i. e. the temporal composition of signals – into account. Finally, Chapter 5.4 addresses stabilization issues of switched linear systems utilizing spatio-temporal symmetry properties.

5.1 Switched Systems and Induced Hybrid Automata

Having dealt with hybrid automata so far, we next discuss a special class of *switched systems* which are closely connected with hybrid dynamical systems in accordance with Definition 2.1.1 and whose dynamics are naturally considered from a shifted point of view. More precisely, a switched system is similar to a dynamical $\mathscr{T}$-system and may be described as a general preform of a hybrid automaton insofar as a chosen switching signal carves out an induced $\mathscr{T}$-system and moreover provides the necessary guard and reset data. In respect thereof, switched systems turn out to be the accurate objects for the remaining part of this thesis. Very broadly speaking, a *switched system* can be thought of as a collection of dynamical systems together with an external *switching rule* (or rather a whole class of admissible switching rules) that chooses (or more descriptively, *switches*) among the systems involved in the course of time.

Switching Signals and Switched Systems

The general concept of switching signals is very simple; a switching signal can be described in terms of map that manages the transitions of a system that lives on a countable set of states. In principle, a switching signal can be defined on both time and the set of states itself. However, we restrict to the case of purely temporal switching.

5.1.1 Definition (Switching Signal). Let Λ be a countable set of discrete states. A *(temporal) switching signal* σ on Λ is a map $\mathbb{T} \to \Lambda$, where $\mathbb{T}$ denotes $\mathbb{R}$ for continuous and $\mathbb{Z}$ for discrete time. $\diamondsuit$

A switching signal $\sigma : \mathbb{T} \to \Lambda$ naturally induces a partition $\{I_k\}_{k\in\mathbb{N}}$ of the time set $\mathbb{T}$: Dependent on the time model $\mathbb{T}$, we have for *continuous* time $\mathbb{T} \subset \mathbb{R}$

$$I_k = [\tau_k, \tau_k') \subset \mathbb{R} \quad \text{with} \quad \tau_k < \tau_k' \quad \text{and} \quad \tau_k' = \tau_{k+1} \tag{5.1}$$

and for *discrete* time $\mathbb{T} \subset \mathbb{Z}$

$$I_k = \{\tau_k, \tau_k + 1, \ldots, \tau_k'\} \subset \mathbb{Z} \quad \text{with} \quad \tau_k \leq \tau_k' \quad \text{and} \quad \tau_k' + 1 = \tau_{k+1}, \tag{5.2}$$

where the time instants $\tau_k \in \mathbb{T}$ correspond to the times of switching. We collect them in the set Δ_σ, i. e. $\Delta_\sigma = \{\tau_k\}_{k\in\mathbb{N}}$. In particular, σ is completely determined by its values on Δ_σ and thus induces a map

$$\widehat{\sigma} : \mathbb{N} \to \Lambda \quad \text{by} \quad \widehat{\sigma}(k) = \sigma(\tau_k).$$

As soon as there is more structure on Λ meaning that Λ is the vertex set of a graph, it is necessary to tackle the question of compatibility of the signal with the discrete structure imposed on Λ by the edges of the graph.

5.1.2 Definition (Weak Admissibility). Let $\mathscr{T} = (\Lambda, \mathcal{E})$ be a directed graph. A switching signal $\sigma : \mathbb{T} \to \Lambda$ is *weakly admissible* if

$$(\widehat{\sigma}(k), \widehat{\sigma}(k+1)) \in \mathcal{E} \quad \text{for all } k \in \mathbb{N}. \tag{5.3}$$

Descriptively speaking, a signal is weakly admissible if it solely switches along edges, i. e. between between vertices that are connected by an edge. In case the set of edges is complete, weak admissibility no longer has a distinguished meaning. For the qualitative study of hybrid dynamics that unfurl subject to a given switching signal σ, not only the sequence of discrete transitions is of importance but undoubtedly also the temporal aspect, meaning the time duration for which the systems stay in the discrete states is of much interest. In order to formally access this temporal data, we introduce the notion of *slowness*.

5.1.3 Definition (Slowness). Let σ be a switching signal and $\beta : \mathcal{E} \to \mathbb{T}_{\geq 0}$ a map. Then σ is said to be *β-slow* if

$$|I_k| = \beta\left(\widehat{\sigma}(k), \widehat{\sigma}(k+1)\right) \quad \text{for all } k \in \mathbb{N}, \tag{5.4}$$

where

$$|I_k| = \tau_k' - \tau_k \in \mathbb{R}_{\geq 0} \quad \text{and} \quad\quad = \mathrm{card}(I_k) \in \mathbb{Z}_{\geq 0} \tag{5.5}$$

in case of continuous and discrete time, respectively. We refer to $\beta = \beta(\sigma)$ as the *slowness (map)* of the signal σ and write

$$\beta_{\sigma,k} = \beta\left(\widehat{\sigma}(k), \widehat{\sigma}(k+1)\right). \tag{5.6}$$

If β is a globally constant function, we identify it with its unique value $\beta \in \mathbb{T}_{\geq 0}$ and β-slowness then simplifies to $\beta_{\sigma,k} = \beta$ for all $k \in \mathbb{N}$. In this case, we refer to σ as being *uniformly β-slow.* $\diamondsuit$

The subsequent definition of switched systems is tailored with regard to the purpose of this thesis and thus may appear unusual at first sight for adepts of switched systems since an underlying switching graph is incorporated; however, in later sections when symmetries come into play, this interpretation of switched systems will turn out to be appropriate. Furthermore, we restrict ourselves to the simplifying assumption that all dynamical systems involved possess the same phase space, and we solely consider purely time-dependent switching.

5.1.4 Definition (Switched System). An n-dimensional *switched system $\mathscr{S}$* is a quintuple $\mathscr{S} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \Omega)$ where

- $\mathscr{T} = (\Lambda, \mathcal{E})$ is the directed *transition* or *switching graph*,
- $\Psi = \{\Psi_\lambda\}_{\lambda \in \Lambda}$ is a collection of n-dimensional dynamical systems $\Psi_\lambda = (\mathcal{D}, \mathcal{F}(\lambda))$ sharing one phase space $\mathcal{D} \subset \mathbb{R}^n$ and
- Ω is the set of all *weakly admissible switching signals* $\sigma : \mathbb{T} \to \Lambda$ with $\mathbb{T}$ denoting time. $\diamondsuit$

Whenever we intend to put a special stress on the presence of the switching graph $\mathscr{T}$, we will speak of a *switched $\mathscr{T}$-system* in the style of dynamical $\mathscr{T}$-systems which will enter the discussion as soon as symmetries enter the scene.

Induced Hybrid Automata and Executions

For a switched system $\mathscr{S}$, in general, the choice of a non-complete transition graph $\mathscr{T}$ corresponds to restricting the set Ω of weakly admissible signals; certainly, $\mathscr{T}$ can be chosen as the complete graph on the vertices Λ such that – from the first – Ω is not subject to any graph-theoretical constraints.

Similar to Definition 4.1.3, we formalize the notion of signal-induced switching graphs in the following.

5.1.5 Definition (Induced Switching Graph). Let $\mathscr{S} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \Omega)$ be a switched system. For a weakly admissible switching signal σ the σ*-induced switching graph* $\mathscr{T}_\sigma = (\Lambda_\sigma, \mathcal{E}_\sigma)$ is given by

$$\Lambda_\sigma = \mathrm{im}(\sigma) \quad \text{and} \quad \mathcal{E}_\sigma = \{(\widehat{\sigma}(k), \widehat{\sigma}(k+1)) \mid k \in \mathbb{N}\} \subset \mathcal{E}. \tag{5.7}$$

For a subgraph $\widetilde{\mathscr{T}}$ of $\mathscr{T}$, we say that a switching signal σ of $\mathscr{S}$ is $\widetilde{\mathscr{T}}$*-exploring* if $\mathscr{T}_\sigma = \widetilde{\mathscr{T}}$. ♢

Having this definition at hand, we see that the slowness β of a switching signal σ is actually a map $\mathcal{E}_\sigma \to \mathbb{R}_{\geq 0}$.

By weak admissibility, the induced switching graph $\mathscr{T}_\sigma$ is a subgraph of $\mathscr{T}$. Intuitively, a switched system together with a weakly admissible switching signal may be considered as a special instance of a hybrid dynamical system along the lines of Definition 2.1.1. However, practically, things are more involved than they seem to be. It is the purpose of this section to discuss these complications and to set up a reasonable groundwork for the appropriate treatment of induced hybrid automata.

We aim at giving a precise description of the induced hybrid systems we are led to when dealing with switched systems and focussing specific switching signals. However, for certain reasons which will be discussed in the following, we have to reduce the set Ω of admissible switching signals by imposing further conditions on the signals. This is due to the fact that – given a β-slow graph-admissible signal σ – the set $\{\beta_{\sigma,k}\}_{k\in\mathbb{N}}$ in general has a greater cardinality than the set $\mathcal{E}_\sigma$. Since the guards of the induced hybrid automaton are created by the numbers $\beta_{\sigma,k}$, this may lead to *different* guards (especially more than one!)

assigned to the *same* edge. So far, if we solely assume admissibility of signals, there is nothing that keeps a signal from doing that. Let $G_e^{\sigma,i}$, $i \in \{1, \dots, n_{\sigma,e}\}$, denote the different guards that arise due to the switching signal σ for the edge $e \in \mathcal{E}_\sigma$ and $n_{\sigma,e}$ their number.

There are different ways to deal with this fact. The first possibility is to create a new vertex and a new edge of the transition graph whenever a different guard is given rise to by the signal. Practically, this means that for an edge $e \in \mathcal{E}_\sigma$, the vertex $\mathfrak{s}e \in \Lambda_\sigma$ is copied $n_{\sigma,e} - 1$ times and the copies $(\mathfrak{s}e)_i$, $2 \leq i \leq n_{\sigma,e}$, are built into the induced graph via the edges e_i, $2 \leq i \leq n_{\sigma,e}$, where $(\mathfrak{s}e)_i = \mathfrak{s}(e_i)$. The obvious consequence of this method is the enlargement of the switching graph, i. e. the σ-induced transition graph is in general no subgraph of the original switching graph. In this case, it is difficult or even impossible to relate the original switching graph to the σ-induced transition graph which is why we will avoid this mechanism. Though if we want to end up with a subgraph of the switching graph, in principle we have to face a *multi-graph* hybrid dynamical system or – which is even worse – a hybrid dynamical system of *non-autonomous* type whose data changes in the course of time. Both cases are not to be considered in this thesis. In order to avoid the occurence of hybrid systems of that kind, we have to phrase conditions keeping us away from them. Thus the second possibility is to form exactly one guard for each edge $e \in \mathcal{E}_\sigma$ on the basis of all guards $G_e^{\sigma,i}$ assigned to e. More concretely, we set

$$G_e = \bigcup_{i=1}^{n_{\sigma,e}} G_e^{\sigma,i} \subset \{\mathfrak{s}e\} \times \mathbb{R}^n \times \mathbb{R}. \tag{5.8}$$

Care is needed when we compare the dynamics of the switched system $\mathscr{S}$ driven by a specific switching signal $\sigma \in \Omega$ to the dynamics of the induced hybrid system $\mathscr{S}_\sigma$. In case of purely state-dependent switching, i. e. the switching signals are of the form $\sigma : D \to \Lambda$, every solution of the switched system $\mathscr{S}$ can be recovered as an execution of the hybrid automaton $\mathscr{S}_\sigma$. However, if the switching is completely time-dependent, things are different. Then the guards are of the form $D_{\widehat{\sigma(k)}} \times \{\beta_{\sigma,k}\}$. Let us consider the case that $n_{\sigma,e} > 1$ for some edge $e \in \mathcal{E}_\sigma$ meaning that the signal σ induces more than one guard assigned to the edge e. Then we find

$$G_e = \bigcup_{i=1}^{n_{\sigma,e}} G_e^{\sigma,i} = \mathcal{D} \times \left\{c_{e_1}, \dots, c_{e_{n_{\sigma,e}}}\right\}.$$

Obviously the resulting hybrid automaton has the same dynamics as the automaton with guards $G_e = \mathcal{D} \times \{c_e\}$ where $c_e = \min_{1 \leq i \leq n_{\sigma,e}} c_{e_i}$. Consequently, executions with temporal data $c_{e_i} > c_e$ cannot occur for this way of realizing the system. With regard to the underlying switching signal σ, this means an immense loss of information.

To overcome or rather avoid such loss of graph-theoretical relations and signal data, we impose additional adequate restrictions on switching signals and thereby introduce a stricter form of admissibility which does not only take into account the transition graph compatibility, but also pays regard to the temporal composition of switching signals. For that purpose, recall that a switching signal σ directly induces a hybrid time trajectory $\tau_\sigma = \{I_k\}_{k \in \mathbb{N}}$ in accordance with Definition 2.2.1.

5.1.6 Definition (Strong Admissibility). Let $\mathscr{S} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \Omega)$ be a switched system. A weakly admissible switching signal $\sigma \in \Omega$ is *strongly admissible* if for every $e \in \mathcal{E}_\sigma$ there is a constant $c_e \in \mathbb{R}_{\geq 0}$ such that

$$|I_k| = c_e \quad \text{for all} \quad k \in \mathbb{N} \quad \text{with} \quad (\widehat{\sigma}(k), \widehat{\sigma}(k+1)) = e, \tag{5.9}$$

where $\tau_\sigma = \{I_k\}_{k \in \mathbb{N}}$ is the σ-induced hybrid time trajectory. We denote by Ω_s the collection of all strongly admissible signals of the switched system $\mathscr{S}$. Clearly, $\Omega_s \subset \Omega$. ♢

For simplicity, we say that a switching signal is *admissible* with respect to a switched system $\mathscr{S}$ if it is both weakly and strongly admissible. Loosely speaking, strong admissibility extends weak admissibility (which is a discrete graph-theoretical condition) to the slowness β of a signal σ and forces the slowness map β to be compatible with the underlying induced switching graph as well. On this basis, we can straightforwardly define induced hybrid automata as follows.

5.1.7 Definition (Induced Hybrid Automaton). Let $\mathscr{S} = (\Lambda, \mathcal{E}, \mathcal{D}, \mathcal{F}, \Omega)$ be a switched system. For a strongly admissible β-slow signal $\sigma \in \Omega$, the *σ-induced hybrid automaton* or *σ-switched system* $\mathscr{S}_\sigma$ is given by the data $(\Lambda_\sigma, \mathcal{E}_\sigma, \mathcal{D}_\sigma, \mathcal{F}_\sigma, \mathcal{G}_\sigma, \mathcal{R}_\sigma)$ where

- $\mathscr{T}_\sigma = (\Lambda_\sigma, \mathcal{E}_\sigma)$ is the *σ-induced switching graph*,

- $\mathcal{D}_\sigma$ is the phase space family $\mathcal{D} = \{D_\lambda\}_{\lambda\in\Lambda_\sigma}$ given by $D_\lambda = \{\lambda\} \times \mathcal{D}$,
- $\mathcal{F}_\sigma$ is the reduced vector field family $\mathcal{F}$ restricted to Λ_σ,
- $\mathcal{G}_\sigma = \{G_e\}_{e\in\mathcal{E}_\sigma}$ is the guard collection induced by σ, i. e. $G_e = D_{\mathfrak{s}e} \times \{c_e\}$
- and $\mathcal{R}_\sigma = \{R_e\}_{e\in\mathcal{E}_\sigma}$ is a family of trivial resets. $\diamondsuit$

It should be pointed out here once more that it is the strong sense of admissibility that makes it possible to interpret an induced hybrid dynamical system $\mathscr{S}_\sigma$ as a hybrid automaton with transition graph $\mathscr{T}_\sigma \sqsubset \mathscr{T}$. The global hybrid phase space D takes the form

$$D = \Lambda \times \mathcal{D}. \tag{5.10}$$

Similar to hybrid automata, we graphically represent a σ-switched system in the way Fig. 5.1 shows. Note that Example 2.1.3 deals with a special hybrid

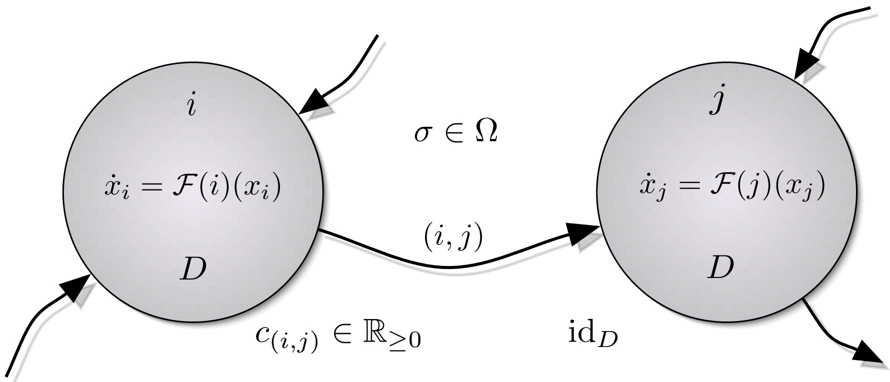


Figure 5.1: Graphical representation of a switched system $\mathscr{S}$ based on Fig. 2.1: Note that the transition graph structure and the constants c_e are induced by the current switching signal $\sigma \in \Omega$.

dynamical system that fits into the framework of switched systems. Translated to the slowness of a switching signal σ, the condition (5.9) comes along as

$$\beta_{\sigma,k} = \beta_{\sigma,k'} \tag{5.11}$$

whenever $\widehat{\sigma}(k) = \widehat{\sigma}(k')$ and $\widehat{\sigma}(k+1) = \widehat{\sigma}(k'+1)$. From now on, each time we encounter a β-slow switching signal σ, we assume that β respects the above condition, i. e. the switching signal σ is strongly admissible.

For a specified switching signal σ, a solution of the σ-switched system $\mathscr{S}_\sigma$ is a collection of local solutions x_k of the dynamical system $\Psi_\lambda = (\mathcal{D}, \mathcal{F}(\lambda))$ orchestrated by the switching signal.

5.1.8 Definition. Let $\mathscr{S}_\sigma$ be a σ-switched system. A solution of $\mathscr{S}_\sigma$ is a family

$$x_\sigma = \{x_k : I_k \to \mathcal{D}\}_{k\in\mathbb{N}} \tag{5.12}$$

such that x_k is a solution of the dynamical systems $\Psi_{\widehat{\sigma}(k)}$ for all $k \in \mathbb{N}$. $\diamondsuit$

One should note with care that even the strong sense of admissibility of switching signals cannot keep an induced hybrid automaton from being non-deterministic:

5.1.9 Note. *Consider a switching graph $\mathscr{T} = (\Lambda, \mathcal{E})$. If there is a vertex $\lambda \in \Lambda$ such that* $\operatorname{card}(\mathcal{O}_{\mathscr{T}}(\lambda)) \geq 2$ *and which is reachable from some $\widetilde{\lambda} \in$ $\mathrm{t}\mathcal{O}_{\mathscr{T}}(\lambda)$ (meaning that there is a directed path from $\widetilde{\lambda}$ to λ), then there exists an admissible switching signal σ such that the σ-switched system $\mathscr{S}_\sigma$ is non-deterministic.*

Since $\operatorname{card}(\mathcal{O}_{\mathscr{T}}(\lambda)) \geq 2$ holds, there are discrete states $\lambda', \lambda'' \in \Lambda$ such that $e' = (\lambda, \lambda')$ and $e'' = (\lambda, \lambda'')$ are edges of $\mathscr{T}$, and since there is path from λ' back to λ, say, there exists an admissible β-slow switching signal $\sigma : \mathbb{R} \to \Lambda$ with $\lambda', \lambda'' \in \Lambda_\sigma$ and $(\widehat{\sigma}(k'), \widehat{\sigma}(k'+1)) = e'$ and $(\widehat{\sigma}(k''), \widehat{\sigma}(k''+1)) = e''$ for some $k', k'' \in \mathbb{N}$ and $\beta_{\sigma,k'} = \beta_{\sigma,k''} = c$. Then the induced switching graph $\mathscr{T}_\sigma$ has $\mathcal{O}_{\mathscr{T}_\sigma}(\lambda) \geq 2$ and has guards $G_{e'} = \mathcal{D} \times \{c\} = G_{e''}$ giving rise to non-determinism.

As opposed to Sections 2.1 and 2.2, we aim to comment on the choice of the time model (continuous or discrete) here and interpret Definition 5.1.4 in the light of these alternatives. In this spirit, $\mathscr{S}$ induces a *discrete*-time or a *continuous*-time switched system dependent on the form of $\mathbb{T}$. For continuous time, we spotlight the relation of switched and hybrid dynamical systems as presented in Section 2.1.

In case of *continuous* time, we can think of a switched system $\mathscr{S}$ as a dynamical system described by a sort of *differential equation*

$$\dot{x}(t) = \mathcal{F}(\sigma(t))(x(t)),$$

where the vector field changes possibly discontinuously with respect to a switching signal $\sigma : \mathbb{R} \to \Lambda$ whose discontinuities Δ_σ correspond to the discrete transitions the system experiences. In this setting, a solution is a family of paths $x_k : I_k \to \mathcal{D}$ fulfilling the corresponding differential equation, more precisely

$$\dot{x}_k(t) = \mathcal{F}\left(\widehat{\sigma}(k)\right)\left(x_k(t)\right) \quad \text{for all} \quad t \in I_k. \tag{5.13}$$

For a *linear* switched system $\mathscr{S}$ with matrix family $\mathcal{F} = \{A_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^{n\times n}$ a solution x_σ with respect to a switching signal σ can be explicitely expressed as follows:

5.1.10 Lemma (cp. [Gök04]). *Let $\mathscr{S}$ be a continuous-time linear switched system and σ an admissible β-slow switching signal with $\sigma(0) = \lambda_0$. A solution of $\mathscr{S}_\sigma$ with initial condition $x_{\sigma(0)}(0) = x_0 \in \mathcal{D}$ then takes the form*

$$x_{\sigma(t)}(t) = \exp\left(A_{\sigma(\tau_N)}\left(t - \sum_{k=0}^{N-1}\beta_{\sigma,k}\right)\right)\prod_{k=0}^{N-1}\exp\left(A_{\sigma(\tau_{N-1-k})}\beta_{\sigma,N-1-k}\right)x_0 \tag{5.14}$$

for $t \in I_N$.

Once a switching signal σ has been chosen, in *discrete* time, a switched system has the form of a *difference equation*

$$x(k+1) = \mathcal{F}(\sigma(k))(x(k))$$

with respect to a switching signal $\sigma : \mathbb{Z} \to \Lambda$. In this case, a solution occurs as a family of orbits $x_k : I_k \to \mathcal{D}$ satisfying

$$x_k(t) = \mathcal{F}\left(\widehat{\sigma}(k)\right)\left(x_k(t)\right) \quad \text{for all} \quad t \in I_k = \left\{\tau_k, \tau_k + 1, \ldots, \tau_k'\right\}. \tag{5.15}$$

Essentially, the fundamental difference between hybrid automata and switched systems lies in the point of view: While – given a hybrid automaton – the central question posed concerns the characterization of the executions that the system accepts, for switched systems, executions are generated by specified external switching signals that are analyzed thereupon. In this sense, the transfer from hybrid dynamical systems to switched systems corresponds to shifting the point of view on the origin of hybrid dynamics.

5.2 Switched Systems and Hybrid Symmetries

In the following, we will discuss switched systems in the presence of hybrid symmetries. We consider a switched system $\mathscr{S} = (\mathscr{T}, \mathcal{D}, \mathcal{F}, \Omega)$. Recall that the choice of an admissible switching signal $\sigma \in \Omega$ determines the guards of the induced hybrid dynamical system, which are given by $\mathcal{G}(e) = \{\mathfrak{s}e\} \times \mathcal{D} \times \{c_e\}$ with switching time $c_e \in \mathbb{R}_{\geq 0}$ corresponding to the discrete transition $e \in \mathcal{E}$. By $\mathscr{S}_\sigma$ we denote the induced switched system and refer to it as *the σ-switched system $\mathscr{S}_\sigma$* which may be considered to be a hybrid dynamical system along the lines of Definition 2.1.1. Nonetheless, *without* specifying any switching signal, we can formally interpret a switched system in terms of a *dynamical $\mathscr{T}$-system* (cp. Definition 3.2.1) and thus deal with its *$\mathscr{T}$-symmetries.* Insofar, with a view to symmetries, the choice of a switching signal establishes the step from $\mathscr{T}$-symmetries to hybrid symmetries. For the switched system $\mathscr{S}$, let $\mathscr{S}_{\mathscr{T}} = (\mathscr{T}, \Theta, \mathcal{F})$ denote the underlying $\mathscr{T}$-system where $\Theta = (\mathcal{D}, \Phi_G)$ provides the group action necessary for the treatment of $\mathscr{T}$-symmetries. Further, let $\mathfrak{S}$ be the accordant group of $\mathscr{T}$-symmetries. Due to the elementary structure of guards and resets, the guard and reset stabilizers simplify to

$$\Sigma_{\mathcal{G}} = \left\{(\pi, g) \in \mathfrak{S} \mid c_{\pi^{-1}(e)} = c_e \ \text{ for all } e \in \mathcal{E}\right\} \quad \text{and} \quad \Sigma_{\mathcal{R}} = \Sigma_{\mathcal{G}}. \tag{5.16}$$

Recall that the slowness β of a switching signal administers the switching times via $\beta(e) = c_e$. Consequently, a particular role is taken up by a signal's slowness since it considerably influences the occuring hybrid symmetries of the induced switched system. So, we arrive at the following straight characterization of hybrid symmetries for switched systems.

5.2.1 Lemma. *Let $\mathscr{S}$ be a switched system, $\sigma \in \Omega$ a β-slow admissible switching signal and $\mathscr{S}_\sigma$ the according σ-switched system. Then, an element $\Upsilon = (\pi, g) \in \operatorname{Aut}(\mathscr{T}_\sigma) \times G$ is a hybrid symmetry of $\mathscr{S}_\sigma$ if and only if it is a $\mathscr{T}$-symmetry of $\mathscr{S}_{\mathscr{T}_\sigma}$ and the graph automorphism π preserves its switching times, i.e. $\beta(\pi^{-1}(e)) = \beta(e)$ for all $e \in \mathcal{E}_\sigma$.*

By virtue of Theorem 3.3.7, the hybrid symmetries $\mathcal{H}_\sigma$ of the σ-switched system $\mathscr{S}_\sigma$ form a group, more precisely a subgroup of $\operatorname{Aut}(\mathscr{T}_\sigma) \times G$. Note that in contrast to hybrid symmetries of a general hybrid system, for switched systems of the type considered here there is only a single group G involved

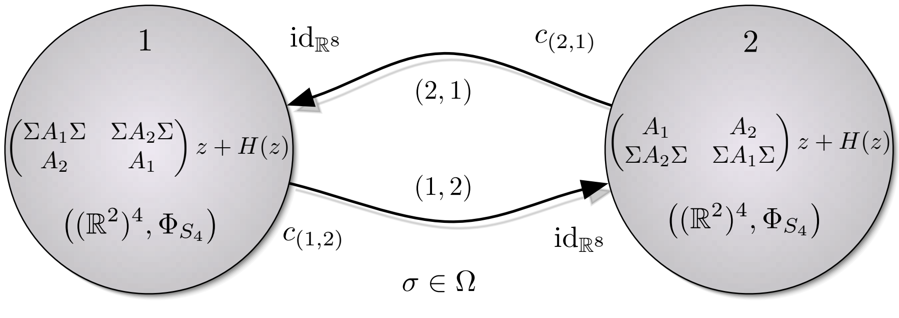


Figure 5.2: Switched system built on the $\mathscr{T}$-system displayed in Figure 3.5.

and, consequently, we only have to deal with a simple $g \in G$ instead of a more complicated $g \in \prod_{\lambda \in \Lambda} G_\lambda$ which keeps us away from compatibility conditions like $\pi^* g = g$ and leaves us on a technically plainer level of thinking.

5.2.2 Example. We consider the continuous-time σ-switched system $\mathscr{S}_\sigma$ illustrated in Fig. 5.2 with switching signal $\sigma : \mathbb{R}_{\geq 0} \to \{1, 2\}$ determined by $\sigma(0) = 1$ and alternating between the discrete states 1 and 2. As discussed in Example 3.2.14, its underlying dynamical $\mathscr{T}$-system $\mathscr{S}_{\mathscr{T}}$ has symmetries $\mathfrak{S} = \{1, \Upsilon_1, \Upsilon_2, \Upsilon_3\}$ with

$$\begin{aligned} \Upsilon_1 &= (\pi, (13)(24), (13)(24))\,, \\ \Upsilon_2 &= (\mathrm{id}_\Lambda, (14)(23), (14)(23))\,, \\ \Upsilon_3 &= (\pi, (12)(34), (12)(34))\,. \end{aligned}$$

Let $\beta : \{(1,2), (2,1)\} \to \mathbb{R}_{>0}$ be the slowness map of the signal σ, i.e. $\beta(1,2) = c_{(1,2)}$ and $\beta(2,1) = c_{(2,1)}$. In case $\beta(1,2) = \beta(2,1)$, the hybrid symmetries of $\mathscr{S}_\sigma$ are $\mathcal{H} = \mathfrak{S}$. Otherwise, if $\beta(1,2) \neq \beta(2,1)$ holds, the hybrid symmetries of $\mathscr{S}_\sigma$ reduce to $\mathcal{H} = \langle \Upsilon_2 \rangle \cong \mathbb{Z}_2$. $\diamondsuit$

For a given switched system $\mathscr{S}$ with underlying dynamical $\mathscr{T}$-system $\mathscr{S}_{\mathscr{T}}$ exhibiting $\mathscr{T}$-symmetries $\mathfrak{S}$, we say that a switching signal $\sigma \in \Omega$ is *maximally symmetry-supporting* if $\mathscr{S}_\sigma$ features the hybrid symmetries $\mathcal{H}_\sigma = \mathfrak{S}$. Lemma 5.2.1 as well as Example 5.2.2 already hint at the meaning of slowness with regard to maximally symmetry-supporting signals. Let $\mathscr{T}^* = (\Lambda^*, \mathcal{E}^*)$ be a subgraph of the switching graph. We call an admissible switching signal σ *$\mathscr{T}^*$-exploring* if $\Lambda_\sigma = \Lambda^*$ or equivalently $\mathscr{T}_\sigma = \mathscr{T}^*$.

5.2.3 Corollary. *Let $\mathscr{S}$ be a switched system with underlying $\mathscr{T}$-system $\mathscr{S}_{\mathscr{T}}$ and $\mathscr{T}$-symmetries $\mathfrak{S}$. Then a $\mathscr{T}$-exploring admissible switching signal σ is maximally symmetry-supporting if and only if its slowness β is $\widehat{\mathfrak{S}}$-invariant.*

Proof. Consider a $\mathscr{T}$-exploring admissible switching signal σ of $\mathscr{S}$ meaning that $\mathscr{T}_\sigma = \mathscr{T}$, in particular $\mathcal{E}_\sigma = \mathcal{E}$. Assume that its slowness map $\beta : \mathcal{E} \to \mathbb{R}_{\geq 0}$ is $\widehat{\mathfrak{S}}$-invariant, i.e. $\beta \circ \pi^{-1} = \beta$ for all $\pi \in \widehat{\mathfrak{S}}$. Especially, for $e \in \mathcal{E}$, we have

$$c_{\pi^{-1}(e)} = \beta(\pi^{-1}(e)) = \beta(e) = c_e$$

for all $\pi \in \widehat{\mathfrak{S}}$. Since this identity holds for every $e \in \mathcal{E}$, we find that $\mathcal{H}_\sigma = \mathfrak{S}$ and, thus, σ is maximally symmetry-supporting. ∎

Note that $\widehat{\mathfrak{S}}$-invariance of β translates to the constancy of β on $\widehat{\mathfrak{S}}$-orbits of edges $e \in \mathcal{E}_\sigma$. Consider a switched system $\mathscr{S}$ with $\mathscr{T}$-symmetries $\mathfrak{S}$. When choosing a switching signal σ the question arises under which conditions $\mathscr{T}$-symmetries of $\mathscr{S}$ induce hybrid symmetries of $\mathscr{S}_\sigma$. This is to ask for the preservation of $\mathscr{T}$-symmetries under the influence of a switching signal. For the case of switched systems, there is a comparably easy and satisfying answer to this question since we are free from compatibility questions for g, as mentioned above.

5.2.4 Proposition. *Let $\mathscr{S}$ be a switched system with $\mathscr{T}$-symmetries $\mathfrak{S}$ and σ a β-slow admissible switching signal. Then a $\mathscr{T}$-symmetry $\Upsilon = (\pi, g) \in \mathfrak{S}$ induces a hybrid symmetry $\Upsilon_\sigma \in \mathcal{H}_\sigma$ of $\mathscr{S}_\sigma$ if and only if π restricts to a symmetry of $\mathscr{T}_\sigma$, i.e. $\pi_\sigma = \pi|_{\Lambda_\sigma} \in \operatorname{Aut}(\mathscr{T}_\sigma)$, and $\beta_{\pi_\sigma^{-1}\circ\sigma,k} = \beta_{\sigma,k}$ for all $k \in \mathbb{Z}$.*

Proof. Suppose that $\Upsilon = (\pi, g) \in \mathfrak{S}$ is a $\mathscr{T}$-symmetry of $\mathscr{S}$ such that the restriction $\pi_\sigma = \pi|_{\Lambda_\sigma}$ is a graph automorphism of the σ-induced switching graph. For $\Upsilon_\sigma = (\pi_\sigma, g)$ we then see

$$\Upsilon_\sigma \mathcal{F}(\lambda) \;=\; g \circ \mathcal{F}(\pi_\sigma^{-1}(\lambda)) \circ g^{-1} = \mathcal{F}(\lambda)$$

for all $\lambda \in \Lambda_\sigma$ since Υ is a $\mathscr{T}$-symmetry of $\mathscr{S}$. If additionally $c_{\pi^{-1}\left(\widehat{\sigma}(k),\widehat{\sigma}(k+1)\right)} = \beta_{\pi_\sigma^{-1}\circ\sigma,k} = \beta_{\sigma,k} = c_{\left(\widehat{\sigma}(k),\widehat{\sigma}(k+1)\right)}$ holds for all $k \in \mathbb{Z}$, Lemma 5.2.1 ensures that Υ_σ is a hybrid symmetry of $\mathscr{S}_\sigma$. The converse direction is clear by definition. ∎

5.3 Symmetry Switching

We now turn to the formal definition and analysis of *symmetry switching*. The idea is to generate switching signals from the symmetries of a given switched system $\mathscr{S}$. Descriptively speaking, after having chosen a $\mathscr{T}$-symmetry $\Upsilon = (\pi, g)$ of $\mathscr{S}$, orbital switching is a way of running through the orbit generated by the automorphism π starting in a specified vertex λ_0. We particularly examine the hybrid automata induced by such symmetry-related switching signals.

5.3.1 Definition (Orbital Switching). Let $\mathscr{S}$ be a switched system with underlying dynamical $\mathscr{T}$-system $\mathscr{S}_{\mathscr{T}}$ admitting the $\mathscr{T}$-symmetry group $\mathfrak{S}$. For $\pi \in \widehat{\mathfrak{S}}$, a switching signal $\sigma : \mathbb{T} \to \Lambda$ is called π-*orbital* if it is of the form

$$\widehat{\sigma}(k) = \pi^{-k}(\lambda_0) \quad \text{for all} \quad k \in \mathbb{N}, \tag{5.17}$$

where λ_0 is the initial discrete state. $\diamondsuit$

The dependence on the initial state $\lambda_0 \in \Lambda$ and the generating element $\pi \in \widehat{\mathfrak{S}}$ will be indicated as $\sigma = \sigma_\pi^{\lambda_0}$, where necessary. To begin with, we address the issue of admissibility for orbital switching signals.

5.3.2 Lemma. *Let $\mathscr{S}$ be a switched system with $\mathscr{T}$-symmetries $\mathfrak{S}$ and $\pi \in \widehat{\mathfrak{S}}$. Then the π-orbital switching signal $\sigma_\pi^{\lambda_0}$ is admissible if and only if $e_0^\pi = (\lambda_0, \pi^{-1}(\lambda_0))$ is an edge of $\mathscr{T}$.*

Proof. If e_0^π is an edge of the switching graph $\mathscr{T}$, then so is $\pi^{-k}(e_0^\pi)$ for all $k \in \mathbb{N}$, because π is a graph automorphism of $\mathscr{T}$. Moreover, we have

$$\mathcal{E} \ni \pi^{-k}(e_0^\pi) = \left(\pi^{-k}(\lambda_0), \pi^{-(k+1)}(\lambda_0)\right) = \left(\sigma_\pi^{\lambda_0}(\tau_k), \sigma_\pi^{\lambda_0}(\tau_{k+1})\right).$$

Hence, $\sigma_\pi^{\lambda_0}$ is admissible if and only if $e_0^\pi \in \mathcal{E}$. ∎

It is clear from the definition that orbital switching belongs to the class of periodic switching signals. The period can be obtained from the slowness β of the signal as shown below.

5.3.3 Lemma. *For $\pi \in \widehat{\mathfrak{S}}$, β-slow π-orbital switching π is periodic with hybrid period $P_\sigma = (N_\sigma, T_\sigma)$, where*

$$N_\sigma = \operatorname{ord}(\pi) \qquad \textit{and} \qquad T_\sigma = \sum_{k=0}^{\operatorname{ord}(\pi)-1} \beta_{\sigma,k}. \tag{5.18}$$

The parametrized slowness map $\beta_\sigma : \mathbb{Z} \to \mathbb{R}$ is N_σ-periodic, i. e. $\beta_{\sigma,k} = \beta_{\sigma,k+N_\sigma}$. Especially, for uniformly β-slow π-orbital switching σ, one has

$$T_\sigma = \operatorname{ord}(\pi)\beta. \tag{5.19}$$

Proof. Let $\sigma = \sigma_\pi^{\lambda_0}$ be a β-slow π-orbital switching signal. Set $N_\sigma = \operatorname{ord}(\pi)$. Then we obtain

$$\widehat{\sigma}(k + N_\sigma) = \pi^{-k-\operatorname{ord}(\pi)}(\lambda_0) = \pi^{-k}(\lambda_0) = \widehat{\sigma}(k)$$

for the discrete part $\widehat{\sigma}$ of σ. Since

$$T_\sigma = \Theta(N_\sigma) = \sum_{k=0}^{\operatorname{ord}(\pi)-1} |I_k| = \sum_{k=0}^{\operatorname{ord}(\pi)-1} \beta_{\sigma,k},$$

we know that $\sigma\left(t + T_\sigma\right) = \sigma(t)$ for all $t \in \mathbb{T}$. For the slowness β_σ one has

$$\begin{aligned} \beta_\sigma(k + N_\sigma) &= \beta\left(\widehat{\sigma}(k + N_\sigma), \widehat{\sigma}(k + N_\sigma + 1)\right) \\ &= \beta\left(\widehat{\sigma}(k), \widehat{\sigma}(k+1)\right) \\ &= \beta_\sigma(k) \end{aligned}$$

by N_σ-periodicity of $\widehat{\sigma}$. Thus β_σ is of period N_σ. If σ is uniformly β-slow, then $\beta_{\sigma,k} = \beta$ for all $k \in \mathbb{N}$ which implies $T_\sigma = \operatorname{ord}(\pi)\beta$. ∎

5.3.4 Remark. However, note carefully that P_σ is generally *not minimal* as a period of the signal $\sigma = \sigma_\pi^{\lambda_0}$. The minimal period - in general - depends on both, the order of π and its relation to the initial discrete state λ_0. This is due to the fact that the order N of a cyclic point $\lambda_0 \in \Lambda$ of a graph automorphism π may be smaller than the order of π as a group element, i. e. $N < N_\sigma$. For instance, if $\lambda \in \operatorname{Fix}_{\mathscr{T}}\left(\langle\pi\rangle\right)$ then $N = 1$ since $\widehat{\sigma}(1) = \pi^{-1}(\lambda_0) = \lambda_0 = \widehat{\sigma}(0)$. Also, N_σ does not have to be minimal as a period of β_σ. ♢

We will come back to this fact a little later. Beforehand, we will address some basic properties of orbital switching. Let Γ_π denote the cyclic group generated by a graph automorphism $\pi \in \widehat{\mathfrak{S}}$, i. e. $\Gamma_\pi = \langle\pi\rangle$.

5.3.5 Lemma. *Let $\mathscr{S}$ be a switched system with $\mathscr{T}$-symmetries $\mathfrak{S}$. Let $\Upsilon = (\pi, g) \in \mathfrak{S}$ and $\lambda_0 \in \Lambda$ such that the β-slow π-orbital switching signal $\sigma = \sigma_\pi^{\lambda_0}$ is admissible.*

(a) *The image $\Lambda_\sigma \subset \Lambda$ of σ is Γ_π-invariant. The same is true for $\mathcal{E}_\sigma$, i. e. the σ-induced switching graph $\mathscr{T}_\sigma$ is Γ_π-invariant.*

(b) *For $\lambda \in \Gamma_\pi \lambda_0$, the β-slow switching signals $\sigma_\pi^{\lambda_0}$ and σ_π^{λ} differ by a time shift, i. e. there exists $\theta_\lambda^{(k)} \in \mathbb{T}$ such that*

$$\sigma_\pi^{\lambda}(t) = \sigma_\pi^{\lambda_0}(t + \theta_\lambda^{(k)}). \tag{5.20}$$

Proof. For the first statement, observe that the image $\Lambda_{\sigma_\pi^{\lambda_0}}$ of the switching signal $\sigma_\pi^{\lambda_0}$ is the orbit of the initial discrete state under the group Γ_π, i. e. $\Lambda_{\sigma_\pi^{\lambda_0}} = \Gamma_\pi \lambda_0$ which is clearly Γ_π-invariant. For $\lambda \in \Gamma_\pi \lambda_0$ and uniformly β-slow switching $\sigma_\pi^{\lambda_0}$ there exists $\kappa_\lambda \in \mathbb{Z}$ such that $\lambda = \pi^{-\kappa_\lambda}(\lambda_0)$. Thus, for $t \in I_k$ we get

$$\sigma_\pi^{\lambda}(t) = \pi^{-k}(\lambda) = \pi^{-k}(\pi^{-\kappa_\lambda}(\lambda_0)) = \pi^{-(k+\kappa_\lambda)}(\lambda_0) = \sigma_\pi^{\lambda_0}(t + \theta_\lambda^{(k)})$$

with

$$\theta_\lambda^{(k)} = \sum_{l=k}^{k+\kappa_\lambda - 1} \beta_{\sigma,l}.$$

Hence, for initial states in the orbit $\Gamma_\pi \lambda_0$, the corresponding switching signals σ_π^{λ} and $\sigma_\pi^{\lambda_0}$ can be transformed into each other by shifting time. ∎

In the following, we consider orbital switching signals induced by conjugated graph symmetries and study their relationship to each other. Here, we make use of the standard notation $g^h = h^{-1}gh$ with $h \in H \leq G$ and $g \in G$ for conjugation with respect to a group G.

5.3.6 Proposition. *Let $\mathscr{S}$ be a switched system with hybrid symmetries $\mathfrak{S}$. Let $\Upsilon = (\pi, g) \in \mathfrak{S}$ and $\lambda_0 \in \Lambda$ such that the β-slow π-orbital switching signal $\sigma = \sigma_\pi^{\lambda_0}$ is admissible.*

(a) *For each $\nu \in \widehat{\mathfrak{S}}$, orbital switching $\sigma_\pi^{\lambda_0}$ relates to conjugation by ν as follows:*

$$\sigma_{\pi^\nu}^{\lambda_0} = \nu^{-1} \circ \sigma_\pi^{\nu(\lambda_0)}. \tag{5.21}$$

(b) *If $\nu \in N_{\widehat{\mathfrak{S}}}(\Gamma_\pi)$, then π^ν-orbital switching either corresponds to π-orbital or to π^{-1}-orbital switching, i. e. ν determines the direction of time.*

(c) *If ν and π commute, π-orbital and π^ν-orbital switching coincide.*

Proof. By definition of π-orbital switching for $\nu \in \widehat{\mathfrak{S}}$ and $t \in I_k$ we compute:

$$\begin{aligned}\sigma_{\pi^\nu}^{\lambda_0}(t) &= \left(\nu^{-1}\pi\nu\right)^{-k}(\lambda_0)\\ &= \left(\nu^{-1}\pi^{-1}\nu\right)^{k}(\lambda_0)\\ &= \left(\nu^{-1}\pi^{-k}\nu\right)(\lambda_0) = \nu^{-1}\sigma_\pi^{\nu(\lambda_0)}(t),\end{aligned}$$

already yielding (a). Now, with a view to (b), let $\nu \in N_{\widehat{\mathfrak{S}}}(\Gamma_\pi)$ which is equivalent to $\nu^{-1}\Gamma_\pi\nu = \Gamma_\pi$. Since $(\nu^{-1}\pi\nu)^k = \nu^{-1}\pi^k\nu$, we have $\mathrm{ord}(\nu^{-1}\pi\nu) = \mathrm{ord}(\pi)$ and

$$\Gamma_{\pi^\nu} = \left\langle \nu^{-1}\pi\nu \right\rangle = \nu^{-1}\langle\pi\rangle\nu = \nu^{-1}\Gamma_\pi\nu = \Gamma_\pi. \tag{5.22}$$

In fact, π and its inverse $\pi^{-1} = \pi^{\mathrm{ord}(\pi)-1}$ are the only elements in Γ_π of order $\mathrm{ord}(\Gamma_\pi)$. This necessarily forces $\pi^\nu \in \{\pi, \pi^{-1}\}$. In case, $\pi^\nu = \pi$, which is equivalent to the case that π and ν commute, $\pi\nu = \nu\pi$, we have $\sigma_{\pi^\nu}^{\lambda_0} \equiv \sigma_\pi^{\lambda_0}$ proving (c). Otherwise, given $\pi^\nu = \pi^{-1}$, we end up with $\sigma_{\pi^\nu}^{\lambda_0} \equiv \sigma_{\pi^{-1}}^{\lambda_0}$. Observing that $\sigma_{\pi^{-1}}^{\lambda_0}(t) = \sigma_\pi^{\lambda_0}(-t)$, we eventually see that conjugation by $\nu \in N_{\widehat{\mathfrak{S}}}(\Gamma_\pi)$ solely affects the direction of time verifying (b). This completes the proof. ■

Next, we aim to address the hybrid automata induced by orbital switching. By Lemma 5.3.3, we know that β-slow π-orbital switching $\sigma = \sigma_\pi^{\lambda_0}$ is periodic with some hybrid period $P_\sigma = (N_\sigma, T_\sigma)$ and that its σ-parametrized slowness map

$$\beta_\sigma : \mathbb{Z} \to \mathbb{R}, \quad \beta_\sigma(k) = \beta\left(\widehat{\sigma}(k), \widehat{\sigma}(k+1)\right) = \beta_{\sigma,k}$$

is N_σ-periodic itself. However, as indicated before, in general there is no reason for N_σ to be minimal as a period of β_σ. In case, β_σ exhibits even richer temporal symmetries meaning that N_σ is not minimal as a period of β_σ providing us with $N \in \mathbb{N}$ such that $N < N_\sigma$ and $\beta_\sigma(k+N) = \beta_\sigma(k)$ for all $k \in \mathbb{Z}$, we may ask for the consequences on the symmetry properties of the σ-switched system $\mathscr{S}_\sigma$. The following proposition provides information on the relation between periodicity of the slowness β_σ and hybrid symmetry properties of the σ-switched system $\mathscr{S}_\sigma$.

5.3.7 Proposition. *Let $\mathscr{S}$ be a switched system with $\mathscr{T}$-symmetries $\mathfrak{S}$. Furthermore, let $\Upsilon = (\pi, g) \in \mathfrak{S}$ be a $\mathscr{T}$-symmetry and $\sigma = \sigma_\pi^{\lambda_0}$ a β-slow π-orbital switching signal with initial discrete state $\lambda_0 \in \Lambda$ such that $e_0^\pi = (\lambda_0, \pi^{-1}(\lambda_0)) \in$*

$\mathcal{E}$. If β_σ is periodic with period $N \leq N_\sigma$, then Υ induces the hybrid symmetry $\Upsilon_\sigma^N \in \mathcal{H}_\sigma$ where $\Upsilon_\sigma = (\pi|_{\Lambda_\sigma}, g)$, and $(\Upsilon_\sigma^N, N) \in \mathcal{H}_\sigma \times \mathbb{Z}_{\mathrm{ord}(\Upsilon)}$ is a hybrid spatio-temporal symmetry of the σ-switched system $\mathscr{S}_\sigma$.

Proof. First of all, admissibility of σ follows from Lemma 5.3.2 since $e_0^\pi \in \mathcal{E}$. We consider the induced switching sequence $\widehat{\sigma} : \mathbb{Z} \to \Lambda$ and for $k \in \mathbb{Z}$ we compute

$$\pi^N(\widehat{\sigma}(k)) = \pi^N\left(\pi^{-k}(\lambda_0)\right) = \pi^{-(k-N)}(\lambda_0) = \widehat{\sigma}(k-N).$$

Thus, we find

$$\pi^{-N}\left(\widehat{\sigma}(k-N)\right) = \widehat{\sigma}(k) \quad \text{or} \quad \pi^{-N}\left(\widehat{\sigma}(k)\right) = \widehat{\sigma}(k+N) \tag{5.23}$$

for all $k \in \mathbb{Z}$. Due to Lemma 5.3.5, the set Λ_σ is π-invariant and hence π restricts to a bijection $\pi_\sigma = \pi|_{\Lambda_\sigma} : \Lambda_\sigma \to \Lambda_\sigma$. Since we additionally have $\mathcal{E}_\sigma = \Gamma_\pi e_0^\pi$, π_σ turns out to be adjacency-preserving with respect to the σ-induced transition graph $\mathscr{T}_\sigma$. Therefore, π_σ is an automorphism of $\mathscr{T}_\sigma$. For algebraic reasons, $\pi_\sigma^N \in \mathrm{Aut}\,(\mathscr{T}_\sigma)$. The N-periodicity of β_σ now enforces

$$\begin{aligned}
\beta_{\pi^{-N}\circ\sigma,k} &= \beta\left(\pi^{-N}(\widehat{\sigma}(k)), \pi^{-N}(\widehat{\sigma}(k+1))\right) \\
&= \beta\left(\widehat{\sigma}(k+N), \widehat{\sigma}(k+N+1)\right) \\
&= \beta_{\sigma,k+N} \\
&= \beta_{\sigma,k}.
\end{aligned}$$

Applying Proposition 5.2.4, we see that $\Upsilon_\sigma^N = \left(\pi_\sigma^N, g^N\right)$ is a hybrid symmetry of $\mathscr{S}_\sigma$. Moreover, we find

$$\mathcal{F}(\widehat{\sigma}(k+N)) \circ g^{-N} = \mathcal{F}\left(\pi_\sigma^{-N}(\widehat{\sigma}(k))\right) \circ g^{-N} = g^{-N} \circ \mathcal{F}(\widehat{\sigma}(k))$$

due to Eq. (5.23) and the fact that Υ_σ^N is a hybrid symmetry of $\mathscr{S}_\sigma$. Hence, $\left(\Upsilon_\sigma^N, N\right)$ turns out to be a hybrid spatio-temporal symmetry of $\mathscr{S}_\sigma$ in accordance with Definition 4.3.1. ∎

Since orbital switching induces hybrid spatio-temporal symmetries of the induced switched system, the hybrid time-T map with T being the period of the switching signal decomposes in a special way according to Section 4.3 which is the subject of the following proposition.

5.3.8 Proposition. *Let $\mathscr{S} = (\Lambda, \mathcal{E}, \Theta, \mathcal{F}, \Omega)$ be a switched system with $\mathscr{T}$-symme-tries $\mathfrak{S}$. For a $\mathscr{T}$-symmetry $\Upsilon = (\pi, g)$, let σ be an admissible β-slow π-orbital switching signal $\sigma = \sigma_\pi^{\lambda_0}$ with initial discrete state $\lambda_0 \in \Lambda$ and N-periodic slowness β_σ. Then the hybrid time-T_σ map $\Phi^\sigma_{T_\sigma}$ – with $T_\sigma = \Theta(N_\sigma)$ and N_σ the discrete period of σ – is of the form*

$$\Phi^\sigma_{T_\sigma} = \Upsilon_\sigma^{\mathrm{ord}(\pi)} \circ \left(\Upsilon_\sigma^{-N} \circ \Phi^\sigma_{\Theta_\tau(N)}\right)^{\frac{\mathrm{ord}(\pi)}{N}}. \tag{5.24}$$

Proof. First of all, for a $\mathscr{T}$-symmetry $\Upsilon = (\pi, g) \in \mathfrak{S}$ the β-slow π-orbital switching signal σ is periodic with hybrid period $P_\sigma = (N_\sigma, T_\sigma)$ where $N_\sigma = \mathrm{ord}(\pi)$ and $T_\sigma = \sum_{k=0}^{\mathrm{ord}(\pi)-1} \beta_{\sigma,k}$ according to Lemma 5.3.3. Since β_σ is N-periodic, Proposition 5.3.7 provides the fact that $\Upsilon_\sigma^N = (\pi_\sigma^N, g^N)$ is indeed a hybrid symmetry of the σ-switched system $\mathscr{S}_\sigma$. Again by virtue of Proposition 5.3.7, we also know that $\left(\Upsilon_\sigma^N, N\right)$ is a hybrid spatio-temporal symmetry of $\mathscr{S}_\sigma$. Trivially, for switched systems, $g_{\widehat{\sigma}(\cdot)}$ is constant on $\mathbb{Z}$. Thus, all requirements of Theorem 4.3.4 are met and for the time-T_σ map $\Phi^\sigma_{T_\sigma}$ we obtain the decomposition

$$\Phi^\sigma_{T_\sigma} = \left(\Upsilon_\sigma^N\right)^\kappa \circ \left(\Upsilon_\sigma^{-N} \circ \Phi^\sigma_{\Theta_\tau(N)}\right)^\kappa$$

with $\kappa = \frac{N_\sigma}{N}$ already proving the statement. ∎

The finest possible decomposition of the hybrid time-$\Theta(N_\sigma)$ map is obtained in the case where the slowness map β is constant (*uniform slowness*).

5.3.9 Corollary. *Let $\Upsilon = (\pi, g) \in \mathfrak{S}$ be a $\mathscr{T}$-symmetry of the switched system $\mathscr{S}$. For an admissible uniformly β-slow π-orbital switching signal $\sigma = \sigma_\pi^{\lambda_0}$, the σ-switched system $\mathscr{S}_\sigma$ exhibits the hybrid spatio-temporal symmetry $(\Upsilon_\sigma, 1)$ and for the hybrid time-T_σ map $\Phi^\sigma_{T_\sigma}$, one has*

$$\Phi^\sigma_{\beta\mathrm{ord}(\pi)} = \Upsilon_\sigma^{\mathrm{ord}(\pi)} \circ \left(\Upsilon_\sigma^{-1} \circ \Phi^\sigma_{\Theta_\tau(1)}\right)^{\mathrm{ord}(\pi)}. \tag{5.25}$$

Proof. The statement stems from the observation that for uniformly β-slow switching σ the induced slowness map β_σ has period 1. Then, according to Proposition 5.3.7, $(\Upsilon_\sigma, 1)$ is a hybrid spatio-temporal symmetry of $\mathscr{S}_\sigma$ and in accordance with Proposition 5.3.8, $\Phi^\sigma_{T_\sigma}$ can be written as

$$\Phi^\sigma_{\beta\mathrm{ord}(\pi)} = \Upsilon_\sigma^{\mathrm{ord}(\pi)} \circ \left(\Upsilon_\sigma^{-1} \circ \Phi^\sigma_{\Theta_\tau(1)}\right)^{\mathrm{ord}(\pi)},$$

completing the proof. ∎

With these preparations at hand, we turn to stabilization issues of switched linear systems in the subsequent section.

5.4 Stabilization and Symmetry Switching

In this section, in which we are going to address stability of a switched system under orbital switching, we restrict ourselves to *linear* switched systems, i.e. $\mathcal{F}$ is a collection of square matrices. We write $\mathcal{A} = \{A_\lambda\}_{\lambda\in\Lambda} \subset \mathbb{R}^{M\times M}$ for clarity. We assume that all matrices involved are invertible. Recall that a matrix $A \in \mathbb{R}^{M\times M}$ is *Schur stable* if all of its eigenvalues lie strictly inside the unit disc, i.e. $\varrho(A) < 1$ with $\varrho(A)$ denoting the spectral radius of A. A matrix A is called *Hurwitz* if all of its eigenvalues have negative real parts.

Consider a linear switched system $\mathscr{S}$ together with a β-slow switching signal σ. Due to Lemma 5.1.10, a solution of the σ-induced system $\mathscr{S}_\sigma$ is given by

$$x_{\sigma(t)}(t) = \exp\left(A_{\widehat{\sigma}(N)}\left(t - \sum_{k=0}^{N-1}\beta_{\sigma,k}\right)\right)\prod_{k=0}^{N-1}\exp\left(\beta_{\sigma,N-1-k}A_{\widehat{\sigma}(N-1-k)}\right)x_0 \tag{5.26}$$

for $t \in I_N$. Thus, for $N \in \mathbb{N}$ we obtain

$$\widehat{\Phi}^\sigma_{\Theta(N)} = \prod_{k=0}^{N-1}\exp\left(\beta_{\sigma,N-1-k}A_{\widehat{\sigma}(N-1-k)}\right) \tag{5.27}$$

for the *continuous* part of the hybrid time $\Theta(N)$-map $\Phi^\sigma_{T_\sigma}$. Especially, in case the signal σ is periodic with hybrid period $P_\sigma = (N_\sigma, T_\sigma)$, the (continuous part of the) time-T_σ map is

$$\widehat{\Phi}^\sigma_{T_\sigma} = \prod_{k=0}^{N_\sigma-1}\exp\left(\beta_{\sigma,N_\sigma-1-k}A_{\widehat{\sigma}(N_\sigma-1-k)}\right). \tag{5.28}$$

An important question arising for switched systems concerns their asymptotic stability. The following (reformulated) result from [Gök04] addresses the asymptotic stability of a switched system that is subject to a periodic switching signal.

5.4.1 Theorem ([Gök04]). *Let $\mathscr{S}$ be a linear switched system and σ a periodic switching signal with hybrid period $P_\sigma = (N_\sigma, T_\sigma)$. Then the σ-switched* *system $\mathscr{S}_\sigma$ is asymptotically stable if and only if the matrix $\widehat{\Phi}^\sigma_{T_\sigma}$ is Schur stable. Equivalently, $\mathscr{S}_\sigma$ is asymptotically stable if and only if the matrix $M_\sigma = \frac{1}{T_\sigma}\log\widehat{\Phi}^\sigma_{T_\sigma}$ is Hurwitz.*

The proof of this theorem consists in an application of the Floquet theorem, the main theorem of Floquet Theory.

5.4.2 Lemma. *For a linear switched system $\mathscr{S}$ with $\mathscr{T}$-symmetries $\mathfrak{S}$, let $\Upsilon = (\pi, g) \in \mathfrak{S}$ be a $\mathscr{T}$-symmetry and $\sigma = \sigma^{\lambda_0}_\pi$ an admissible β-slow π-orbital switching signal with N-periodic β_σ. Moreover, let $\rho : \widehat{\mathfrak{S}} \to O(M)$ be a representation of $\widehat{\mathfrak{S}} \leq G$. For $\Sigma = \rho(g)$, the continuous part of the time-T_σ map is*

$$\widehat{\Phi}^\sigma_{T_\sigma} = \Sigma^{-\mathrm{ord}(\pi)}\left(\Sigma^N \prod_{k=0}^{N-1} \exp\left(\beta_{\sigma,N-1-k}A_{\widehat{\sigma}(N-1-k)}\right)\right)^{\frac{\mathrm{ord}(\pi)}{N}}. \tag{5.29}$$

For uniformly β-slow π-orbital switching σ, one has

$$\widehat{\Phi}^\sigma_{T_\sigma} = \Sigma^{-\mathrm{ord}(\pi)}\left(\Sigma\exp\left(\beta A_{\lambda_0}\right)\right)^{\mathrm{ord}(\pi)}. \tag{5.30}$$

Proof. By virtue of Propositions 5.3.7 and 5.3.8 we arrive at the identity

$$\Phi^\sigma_{T_\sigma} = \Upsilon^{N_\sigma}_\sigma \circ \left(\Upsilon^{-N}_\sigma \circ \Phi^\sigma_{\Theta(N)}\right)^{\frac{N_\sigma}{N}}$$

for the hybrid time-T_σ map. Via Eq. (5.27) and due to the fact that hybrid symmetries act as $\Upsilon(\lambda, x) = (\pi^{-1}(\lambda), g^{-1}x)$, we indeed obtain

$$\widehat{\Phi}^\sigma_{T_\sigma} = \Sigma^{-\mathrm{ord}(\pi)}\left(\Sigma^N \prod_{k=0}^{N-1} \exp\left(\beta_{\sigma,N-1-k}A_{\widehat{\sigma}(N-1-k)}\right)\right)^{\frac{\mathrm{ord}(\pi)}{N}}$$

proving the first statement. For the second statement, simply note that uniformly β-slow switching corresponds to the case $N = 1$. With $\beta_{\sigma,0} = \beta$ and $\widehat{\sigma}(0) = \lambda_0$ we arrive at the desired statement. ∎

For a β-slow π-orbital switching signal $\sigma^{\lambda_0}_\pi$ and $N \in \mathbb{N}$, let $\widetilde{A}^\beta_{\pi,N}$ be the matrix

$$\widetilde{A}^\beta_{\pi,N} = \prod_{k=0}^{N-1} \exp\left(\beta_\sigma(N-1-k)A_{\pi^{-N+1+k}(\lambda_0)}\right). \tag{5.31}$$

5.4.3 Theorem. *For a linear switched system $\mathscr{S}$, let $\lambda_0 \in \Lambda$ and $\Upsilon = (\pi, g) \in \mathfrak{G}$ such that e_0^{π} and $e_0^{\pi^{-1}}$ are edges of the switching graph $\mathscr{T}$. Let $\sigma = \sigma_{\pi^\nu}^{\lambda_0}$ denote β-slow π^ν-orbital switching $\sigma = \sigma_{\pi^\nu}^{\lambda_0}$ with N-periodic slowness β_σ, $N \leq N_\sigma$. If the matrices $\widetilde{A}_{\pi,N}^{\beta}$ and $\widetilde{A}_{\pi^{-1},N}^{\beta}$ are Schur stable, then $\mathscr{S}$ is asymptotically stable under the switching signal $\sigma_{\pi^\nu}^{\lambda_0}$ for all $\nu \in N_{\widehat{\mathfrak{G}}}(\Gamma_\pi)$.*

Proof. Let ν be an element of the normalizer $N_{\widehat{\mathfrak{G}}}(\Gamma_\pi)$ and $\widetilde{\Upsilon} = (\nu, \widetilde{g})$ an according $\mathscr{T}$-symmetry of $\mathscr{S}$. We consider β-slow π^ν-orbital switching σ with N-periodic slowness β_σ. In accordance with Proposition 5.3.6 (b), we know that $\pi^\nu \in \{\pi, \pi^{-1}\}$, i.e. σ corresponds to π- or π^{-1}-orbital switching. Since $e_0^{\pi}, e_0^{\pi^{-1}} \in \mathcal{E}$, the signal $\sigma_{\pi^\nu}^{\lambda_0}$ is admissible due to Lemma 5.3.2. Let $\rho : \widehat{\mathfrak{G}} \to O(M)$ be a representation of $\widehat{\mathfrak{G}}$. Now, Lemma 5.4.2 applies and with $\rho(\widetilde{g}^{-1} g \widetilde{g}) = \Sigma$ and $\mathrm{ord}(\pi^\nu) = \mathrm{ord}(\pi)$ we obtain

$$\widehat{\Phi}_{T_\sigma}^{\sigma} = \Sigma^{-\mathrm{ord}(\pi)} \left(\Sigma^N \widetilde{A}_{\pi^\nu,N}^{\sigma}\right)^{\frac{\mathrm{ord}(\pi)}{N}} .$$

For the spectral radius of $\widehat{\Phi}_{T_\sigma}^{\sigma}$, we obtain

$$\varrho\left(\widehat{\Phi}_{T_\sigma}^{\sigma}\right) = \varrho\left(\Sigma^{-\mathrm{ord}(\pi)} \left(\Sigma^N \widetilde{A}_{\pi^\nu,N}^{\beta}\right)^{\mathrm{ord}(\pi)}\right) = \varrho\left(\left(\Sigma^N \widetilde{A}_{\pi^\nu,N}^{\beta}\right)^{\mathrm{ord}(\pi)}\right)$$

by invariance of ϱ under multiplication by orthogonal matrices. We can proceed using Gelfand's formula $\varrho(A) = \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}}$ and find

$$\varrho\left(\widehat{\Phi}_{T_\sigma}^{\sigma}\right) \leq \varrho\left(\Sigma^N \widetilde{A}_{\pi^\nu,N}^{\beta}\right)^{\mathrm{ord}(\pi)} = \varrho\left(\widetilde{A}_{\pi^\nu,N}^{\beta}\right)^{\mathrm{ord}(\pi)} .$$

Since $\widetilde{A}_{\pi^\nu,N}^{\beta} = \widetilde{A}_{\pi,N}^{\beta}$ or $= \widetilde{A}_{\pi^{-1},N}^{\beta}$, by assumption we have $\varrho\left(\widetilde{A}_{\pi^\nu,N}^{\beta}\right) < 1$ yielding $\varrho\left(\widehat{\Phi}_{T_\sigma}^{\sigma}\right) < 1$. According to Theorem 5.4.1, $\mathscr{S}$ is asymptotically stable under β-slow π^ν-orbital switching for every $\nu \in N_{\widehat{\mathfrak{G}}}(\Gamma_\pi)$. ∎

Especially for uniformly β-slow orbital switching we obtain the following symmetry-related stability result.

5.4.4 Corollary. *For a linear switched system $\mathscr{S}$, let $\lambda_0 \in \Lambda$ and $\Upsilon = (\pi, g) \in \mathfrak{G}$ such that e_0^{π} and $e_0^{\pi^{-1}}$ are edges of the switching graph $\mathscr{T}$. If there exists $\lambda \in \Lambda_\sigma$ with $\widetilde{A}_{\lambda}^{\beta} = \exp(\beta A_\lambda)$ Schur stable, $\mathscr{S}$ is asymptotically stable under uniformly β-slow π^ν-orbital switching $\sigma = \sigma_{\pi^\nu}^{\lambda_0}$ for all $\nu \in N_{\widehat{\mathfrak{G}}}(\Gamma_\pi)$.*

Proof. For $\lambda \in \Lambda_\sigma = \Gamma_\pi \lambda_0$ there exists $k_\lambda \in \{0, \ldots, \operatorname{ord}(\pi)\}$ such that $\lambda = \pi^{-k_\lambda}(\lambda_0) = \sigma_\pi^{\lambda_0}(k_\lambda)$. Then for the hybrid symmetry $\Upsilon^{k_\lambda} = \left(\pi^{k_\lambda}, g^{k_\lambda}\right)$ one has $A_{\lambda_0} = \Sigma^{k_\lambda} A_\lambda \Sigma^{-k_\lambda}$ and thus $\exp(\beta A_{\lambda_0}) = \Sigma^{k_\lambda} \exp(\beta A_\lambda) \Sigma^{-k_\lambda}$. That is why we have $\varrho(\widetilde{A}_\lambda^\beta) = \varrho(\widetilde{A}_{\lambda_0}^\beta)$. Observe that in this case $\widetilde{A}_{\pi,1}^\beta = \widetilde{A}_{\lambda_0}^\beta = \exp(\beta A_{\lambda_0}) = \widetilde{A}_{\pi^{-1},1}^\beta$ holds. Consequently,

$$\varrho\left(\widetilde{A}_{\pi,1}^\beta\right) = \varrho\left(\widetilde{A}_{\pi^{-1},1}^\beta\right) = \varrho\left(\exp(\beta A_{\lambda_0})\right) = \varrho\left(\exp(\beta A_\lambda)\right) < 1,$$

hence both $\widetilde{A}_{\pi,1}^\beta$ and $\widetilde{A}_{\pi^{-1},1}^\beta$ are Schur stable, and Theorem 5.4.3 yields the desired result. ■

Herewith, we arrive at a stability result which on the one hand accounts for the hybrid symmetries of the switched system and on the other hand takes notice of the temporal organization of the switching signal. It is not surprising that the strongest result occurs for the case of uniform switching. Moreover, we see that stability properties pertain conjugacy classes of orbital switching signals, when mild algebraic requirements are assumed.

Chapter 6

Time-Varying Networks of Dynamical Systems

The general way of analyzing a phenomenon occuring in the real world with respect to mathematics, is to set up a mathematical model, which will then replace the original phenomenon in the course of further examination. However, what sounds so plain and straightforward at first is usually closely accompanied by immense difficulties arising from hard-to-take decisions and severe choices due to missing or unavailable knowledge.

The treatment of non-autonomous dynamical systems plays a major role within the construction of various models for all kinds of real world phenomena. One may even say that everything that surrounds us (be it embedded in nature, linked with technology or characterized by social attributes) non-trivially depends on time: Time directs everything, from the progressional expansion of the universe up to the life story of a cell – reality is definitely *non*-static.

What makes things even more intricate is the impression that reality is superlatively complex. Essentially, reality draws this intricacy from its composition of countless interacting instances, which is mathematically formulated in terms of networked dynamical systems or *coupled cell systems*. Taking care of explicit time-dependence, one ends up with *time-varying dynamical system networks* as general models to adequately describe real world issues. As spatially discrete structures, networks naturally evolve instantaneously, i. e. in discrete manners, while the single systems – communicating via network structures – evolve continuously. It is this matter which a priori classifies time-varying dynamical system networks as *hybrid systems*.

This closing chapter finally centers on such time-varying dynamical system networks and their relation to hybrid dynamical systems from the special viewpoint of symmetries and thus takes on the leading motives discussed at the outset of this thesis. After Chapter 6.1 has introduced the general notion of *coupled cell systems*, a particular class of D_4-symmetric dynamical system networks is discussed in Chapter 6.2. So far, the network architectures involved are temporally fixed and so the systems occuring are classically smooth. The situation fundamentally changes as soon as in Chapter 6.3 the globally symmetric dynamical systems are subjected to periodic forcing affecting the network structure and breaking their spatial symmetries to reduced spatio-temporal symmetries. Thereupon a discretization process lead to the design of a related hybrid dynamical system which is found to possess hybrid symmetries linked to the former spatio-temporal symmetries of the forced system. Ultimately, Chapter 6.4 presents a numerical treatment of this derived hybrid dynamical system adressing stability and applying the afore developed theory.

6.1 Coupled Cell Systems

In the field of dynamical systems, a *coupled cell system* is understood as a dynamical system that is composed of a collection of smaller subsystems that influence each other dynamically. When speaking of coupled cell systems, one has to distinguish between two related but fundamentally different concepts which vary in the way the system's symmetries are perceived. Both of these concepts are essentially due to Martin Golubitsky and Ian Stewart who have published numerous works on this matter in the course of the last decade and have strongly influenced the structural treatment of networked dynamical systems.

The primary understanding of the term *coupled cell system* which will be used in the framework of this thesis classifies a coupled cell system as an equivariant dynamical system in the classical sense, i. e. the (global) symmetries form a group acting on the phase space and the overall vector field is equivariant with respect to this group which is induced by a special fine structure of the system. This notion of coupled cell system is introduced in [GS02].

A further development is brought forth by the experience that global symmetries are not the only structural traits to cause certain dynamical phenomena like the synchrony of subsystems, but a softer kind of symmetry which is of *local* nature suffices for the creation of specifically structured dynamics. Generally, a system that has trivial global symmetry properties may possess immensely rich local symmetry structure. In addition, global symmetry properties are highly sensitive with respect to network perturbations (e. g. adding or deleting edges), while local symmetries are very persistant. The weakening of symmetries from global to local is realized and incorporated into the conception of coupled cell systems in a series of publications starting in 2003 by Golubitsky, Stewart and coworkers. However, as a consequence of this modified notion of symmetry, severe algebraic difficulties arise: As a matter of fact, the collection of local symmetries can no longer be algebraically described with the help of groups, because the restrictions of group theory are broken up by the solely locally compatible symmetries. Instead, these symmetries are found to give rise to a *groupoid* which represents a considerably weaker manifestation of algebraic structure.[1]

A coupled cell system is perceived as a network of dynamical systems in both cases, i. e. a coupled cell system is hybrid in the sense that it combines a *discrete* underlying graph structure determining the coupling architecture of the subsystems with the *continuous* dynamical systems assigned to the vertices. Another important similarity lies in the fact that the symmetries stem from the coupling network which as a graph gathers its symmetry information in its automorphism group or its symmetry groupoid, respectively. Actually, if a coupled cell system exhibits symmetries in the global sense, then the vector field's classical equivariance is equivalent to its equivariance with respect to a certain induced groupoid (see [DS04]).

As mentioned before, we stick to the global way of understanding symmetry and keep in mind the treatment of [GS02] with regard to contents, but formally proceed in the style of [SGP03]. Concisely speaking, this means that we do

[1] It is not surprising that the handling of this sort of structure is more complicated and involves novel technical difficulties and administrative efforts that have not been present before in the global symmetry setting.

not start with a dynamical system $\dot{x} = F(x)$ and formulate properties that characterize it as a coupled cell system, but define a coupled cell system constructively in layers, i. e. beginning with the coupling network as a fundament, we build the system on top of it.

6.1.1 Definition (Coupled Cell System). Let $\mathcal{C} = \{1, \dots, N\}$ be an index set, referred to as the collection of *cells*. A *coupled cell system* is a dynamical system $\dot{x} = F(x)$ whose phase space $X \subset \mathbb{R}^M$ is a Cartesian product

$$X = \prod_{c \in \mathcal{C}} X_c \tag{6.1}$$

equipped with a family $\Pi = \{\pi_c\}_{c \in \mathcal{C}}$ of canonical projections $\pi_c : X \to X_c$. ♢

This definition is quite dim and conceals the actual structure of a coupled cell system to a certain extent. The fundamental idea of coupled cell systems which explicitely distinguishes a coupled cell system from a general dynamical system is its specific fine structure brought out by the product structure of the phase space and the according projections. This makes it possible to interpret a dynamical system as a network of smaller systems – its *cells* – and understand it in terms of these cells. For a trajectory $x(t)$, one obtains the *cell trajectories* $x_c(t) = \pi_c(x(t))$ for $c \in \mathcal{C}$. The cell trajectories then appear as trajectories of the *individual* cells and it is this point of view that allows to address synchrony properties of the dynamics.

With a coupled cell system we can associate a directed graph $\mathcal{G} = (\mathcal{C}, \mathcal{E})$ whose vertices correspond to the cells $c \in \mathcal{C}$ and whose edges $\mathcal{E}$ describe the couplings between the cells. More precisely, the graph $\mathcal{G}$ contains an edge $e = (i, j)$ from cell i to cell j if and only if cell j is influenced by cell i meaning that the map $F_j(x) = \pi_j(F(x))$ depends on $x_i = \pi_i(x)$, $x \in X$. In order to incorporate different types of coupling and different types of cells, we use a *decorated* graph, practically meaning that we use the same symbol for all vertices of the same type and the same arrow for all edges of the same type. This is realized by equivalence relations $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{E}}$ on $\mathcal{C}$ and $\mathcal{E}$, respectively. We cast the preceding considerations into the following definition (cp. [SGP03] or [GS06]).

6.1.2 Definition (Coupled Cell Network). A *coupled cell network* $\mathcal{G}$ is determined by the quadruple $(\mathcal{C}, \sim_{\mathcal{C}}, \mathcal{E}, \sim_{\mathcal{E}})$ where

- $\mathcal{C} = \{1, \ldots, N\}$ is a finite collection of *cells*,
- $\sim_\mathcal{C}$ is an equivalence relation on $\mathcal{C}$ and the equivalence class $[c]_\mathcal{C} \in \mathcal{C}/\sim_\mathcal{C}$ is called the *cell type* of c,
- $\mathcal{E} \subset \mathcal{C} \times \mathcal{C}$ is a finite set of *edges* and
- $\sim_\mathcal{E}$ is an equivalence relation on $\mathcal{E}$ and the equivalence class $[e]_\mathcal{E} \in \mathcal{E}/\sim_\mathcal{E}$ is referred to as the *edge type* of e

such that the following consistency condition is fulfilled:

- Equivalent edges have equivalent sources and targets, i. e. the relation $e_1 \sim_\mathcal{E} e_2$ implies

$$\mathfrak{s}e_1 \sim_\mathcal{C} \mathfrak{s}e_2 \quad \text{and} \quad \mathfrak{t}e_1 \sim_\mathcal{C} \mathfrak{t}e_2. \tag{6.2}$$

Having the notion of a coupled cell network at hand, we have access to the formal perception as well as the description of coupled cell systems as networks of dynamical systems. As a matter of course, a coupled cell network solely describes the discrete part of a coupled cell system and does not give any account of the cells' and their couplings' qualitative meaning. As opposed to Definition 6.1.1 which focuses upon a system of ODEs whose special structural characteristics are not immediately obvious, we follow the converse path and set out from a given coupled cell network and assign the dynamics to the nodes and provide the edges with meaning. For this purpose, we define vector fields that are compatible with the considered coupled cell network. Before doing so, we have to mention the symmetry properties of the coupled cell network and equip the cells with phase spaces. Addressing symmetries, we point out that a coupled cell network $\mathcal{G}$ as a decorated, directed graph comes along with its automorphism group $\mathrm{Aut}(\mathcal{G}) \leq S_{\mathrm{card}(\mathcal{C})}$; note that the bigger $\mathcal{C}/\sim_\mathcal{C}$ is, the less symmetric is $\mathcal{G}$. We now assign a phase space X_c to each cell $c \in \mathcal{C}$ such that the relation $c \sim_\mathcal{C} d$ implies $X_c = X_d$ and define the *total phase space* as the product space $X = \prod_{c\in\mathcal{C}} X_c$. Here, for simplicity, we restrict to finite-dimensional real vector spaces. For a cell $c \in \mathcal{C}$, the *input set* $I(c)$ of c is defined as the set

$$I(c) = \{e \in \mathcal{E} \mid \mathfrak{t}e = c\} \subset \mathcal{E}. \tag{6.3}$$

For a finite ordered set of n cells $\mathcal{C}_n = \{c_1, \ldots, c_n\}$, set

$$X_{\mathcal{C}_n} = \prod_{i=1}^{n} X_{c_i} \quad \text{and} \quad x_{\mathcal{C}_n} = (x_{c_1}, \ldots, x_{c_n}) \in X_{\mathcal{C}_n}.$$

These preparations enable us to define the class of admissible vector fields for a given coupled cell network $\mathcal{G}$.

6.1.3 Definition (Admissible Vector Fields). Let $\mathcal{G} = (\mathcal{C}, \sim_{\mathcal{C}}, \mathcal{E}, \sim_{\mathcal{E}})$ be a coupled cell network with compatible phase space family $\{X_c\}_{c\in\mathcal{C}}$. A vector field $F : X \to X$ is called $\mathcal{G}$-*admissible* if it meets the following two conditions:

- For all $c \in \mathcal{C}$, there exists a map $\widehat{f}_c : X_c \times X_{\mathfrak{s}(I(c))} \to X_c$ such that
$$f_c(x) = \widehat{f}_c(x_c, x_{\mathfrak{s}I(c)}) \tag{6.4}$$
for all $x \in X$, where f_c is the component $\pi_c \circ f$ according to cell c.
- F is equivariant with respect to $\operatorname{Aut}(\mathcal{G})$. ♢

The first condition is termed the *domain condition* and ensures that admissible vector fields reflect the coupling architecture prescribed by the coupled cell network and the second one is simply referred to as the *equivariance condition*. We study an exemplary coupled cell system in the following section.

6.2 A Particular Class of Coupled Cell Systems

This section concentrates on the structural composition of a special class of coupled cell systems examining the according equations in the light of this structure. It is therefore of illustrative character. We consider the coupled cell network $\mathcal{G}$ determined by the cells

$$\mathcal{C} = \{1, 2, 3, 4\}$$

and the edges

$$\mathcal{E} = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3), (4,1), (1,4)\}$$

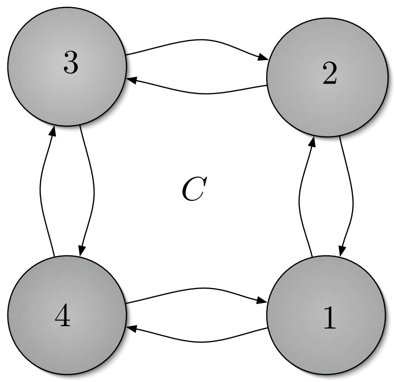


Figure 6.1: D_4-symmetric coupled cell network

such that all cells and edges are of the same type defining the equivalence relations $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{E}}$. This data gives rise to a decorated directed graph $\mathcal{G}$ with dihedral symmetry, i. e. $\mathrm{Aut}\,(\mathcal{G}) = D_4$, as shown in Fig. 6.1. We assign phase spaces to the cells in a consistent manner, i. e. $X_1 = X_2 = X_3 = X_4 = X$ since all cells are equivalent. A general admissible vector field $\mathcal{F} : X^4 \to X^4$ takes the form

$$\mathcal{F}(x_1, x_2, x_3, x_4) = \begin{pmatrix} g_1(x_1, x_2, x_4) \\ g_2(x_2, x_3, x_1) \\ g_3(x_3, x_2, x_4) \\ g_4(x_4, x_1, x_3) \end{pmatrix}, \tag{6.5}$$

where $g_1 = g_2 = g_3 = g_4 = g$ holds with $g : X^3 \to X$, which is due to identical cells and couplings. In order to see that, we observe that for the input sets, we have

$$\begin{aligned} I(1) &= \{(2,1),(4,1)\}\,, \\ I(2) &= \{(1,2),(3,2)\}\,, \\ I(3) &= \{(2,3),(4,3)\}\,, \\ I(4) &= \{(1,4),(3,4)\}\,, \end{aligned}$$

and thus the domain condition (6.4) is easily seen to be fulfilled. Additionally, we notice D_4-equivariance of $\mathcal{F}$. For the phase spaces, we select $X_i = \mathbb{R}^n$. With the choice of diffusive coupling, the resulting global dynamical system is determined by the equations

$$\begin{aligned}
\dot{x}_1 &= f(x_1) + \zeta D\left((x_2 - x_1) + (x_4 - x_1)\right) \\
\dot{x}_2 &= f(x_2) + \zeta D\left((x_3 - x_2) + (x_1 - x_2)\right) \\
\dot{x}_3 &= f(x_3) + \zeta D\left((x_4 - x_3) + (x_2 - x_3)\right) \\
\dot{x}_4 &= f(x_4) + \zeta D\left((x_1 - x_4) + (x_3 - x_4)\right)
\end{aligned} \tag{6.6}$$

with a parameter $\zeta \in \mathbb{R}$ governing the coupling strength, a diffusion matrix $D \in \mathbb{R}^{n \times n}$ and a vector field $f : X \to X$ determining the internal dynamics of the cells. See Fig. 6.2 for an illustration of the considered coupled cell system. In a more compact way, the system may also be written as

$$\begin{aligned}
\dot{x}_i &= f(x_i) + \zeta D\left((x_{i+1} - x_i) + (x_{i-1} - x_i)\right) \\
&= f(x_i) + \zeta D\left(x_{i+1} - 2x_i + x_{i-1}\right), \qquad i = 1, \ldots, 4
\end{aligned}$$

where the boundary conditions $x_5 = x_1$ and $x_0 = x_4$ have to be noted carefully. Using the Laplacian $L = \operatorname{diag}(d) - C$ of the underlying coupling network, where the *connection* or *adjacency* matrix $C = \{C_{ij}\}_{i,j=1,\ldots,4} \in \mathbb{R}^{4\times 4}$ is given by

$$C_{ij} = \begin{cases} 1 & \text{if cell } j \text{ is coupled to cell } i \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

and $d \in \mathbb{N}^4$ is the vector of degrees with components $d_i = \deg(i) = \sum_{j \neq i} C_{ij}$, we can write

$$\dot{x} = \mathcal{F}(x) = F(x) - (L \otimes \mathbb{I}_n)\, H(x) \tag{6.8}$$

with global vector field

$$F : \mathbb{R}^{4n} \to \mathbb{R}^{4n}, \quad F(x_1, x_2, x_3, x_4) = \left(f(x_1), f(x_2), f(x_3), f(x_4)\right)^T$$

and coupling function

$$H(x; \zeta, D) = \left(h(x_1), h(x_2), h(x_3), h(x_4)\right)^T \quad \text{with} \quad h(x_i) = h(x_i; \zeta, D) = \zeta D x_i.$$

Here, $\otimes$ denotes the Kronecker product of matrices and in this case of dihedral network symmetries, the adjacency and the Laplacian matrix take the form

$$C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{6.9}$$

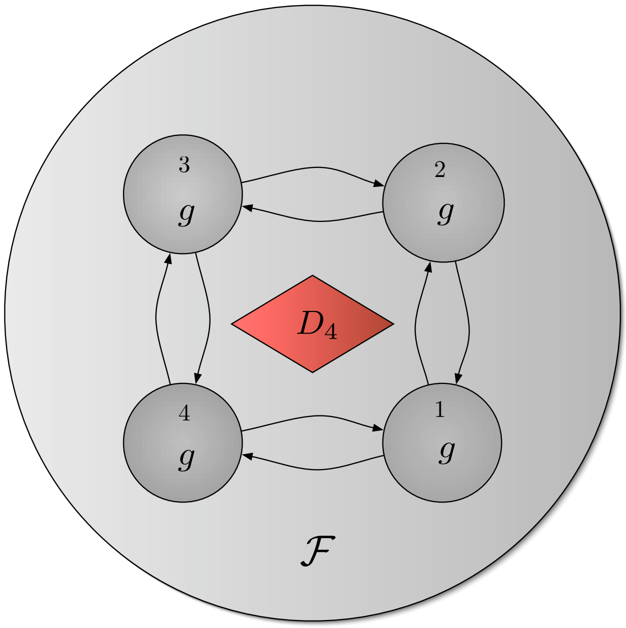


Figure 6.2: Dynamical system network consisting of four identical subsystems (*cells*) and identical coupling: The global system exhibits D_4-symmetry (apart from further symmetries featured by the cells) owing to the dihedral symmetry of the coupling graph.

and

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \tag{6.10}$$

respectively. With Eq. (6.8), we now have a formulation for the system at hand which allows for the direct access of the system's internal coupling. This will be of decisive advantage for the subsequent considerations.

6.3 From Periodically Driven Coupled Cell Systems to Orbitally Switched Systems

This section is designed to shed light on a specific procedure that gives rise to orbitally switched systems on the grounds of coupled cell systems that are subject to external periodic forcing. It builds on the example of Chapter 6.2. The approach is as follows: A smooth periodic network perturbation is introduced to the original system yielding a non-autonomous vector field that exhibits periodicity in time and reveals spatio-temporal symmetry properties. The external forcing due to the network perturbation may be interpreted as an instance of *forced symmetry breaking.* Thereupon, the resulting vector field family is discretized in an adequate way and the resulting vector fields are connected on the basis of a natural transition graph structure giving rise to a switched system with non-trivial hybrid symmetries.

We consider the coupled cell system (6.8) with architecture as illustrated in Fig. 6.2. Let $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a sufficiently smooth, periodic function with minimal period $T > 0$ such that

$$\psi\left(\frac{5T}{8}\right) = 0 \quad \text{and} \quad \psi\left(\frac{T}{8}\right) = \psi\left(\frac{3T}{8}\right) = \psi\left(\frac{7T}{8}\right) = 1. \tag{6.11}$$

See Fig. 6.3 for a plot of an exemplary function ψ.

We modify the system

$$\dot{x} = \mathcal{F}(x) = F(x) - (L \otimes \mathbb{I}_n)H(x)$$

by forcing the adjacency matrix C in the following manner:

$$C(t) = \begin{pmatrix} 0 & \psi\left(t + \frac{3T}{4}\right) & 0 & \psi(t) \\ \psi\left(t + \frac{3T}{4}\right) & 0 & \psi\left(t + \frac{T}{2}\right) & 0 \\ 0 & \psi\left(t + \frac{T}{2}\right) & 0 & \psi\left(t + \frac{T}{4}\right) \\ \psi(t) & 0 & \psi\left(t + \frac{T}{4}\right) & 0 \end{pmatrix}. \tag{6.12}$$

As a consequence, both the degree map $\deg : \mathcal{C} \to \mathbb{Z}$ and the Laplacian matrix

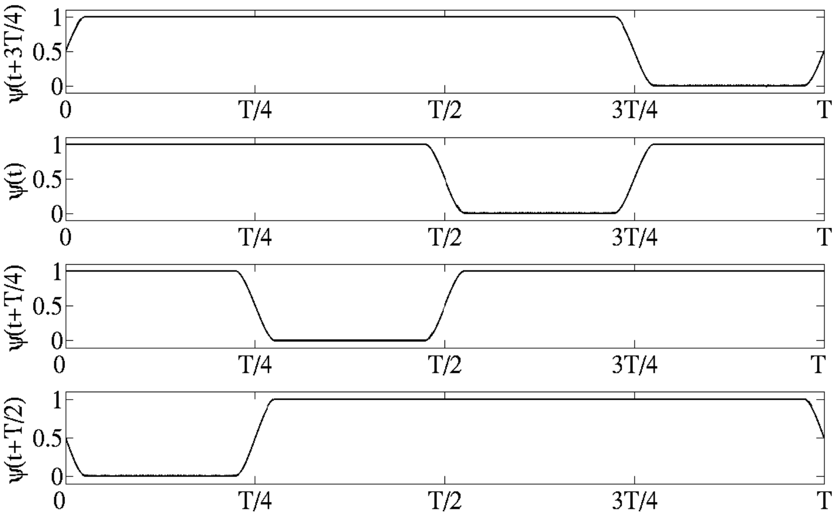


Figure 6.3: Smoothly time-variant coupling: The time-varying coupling of a dynamical system network is realized by phase shifts of a smooth periodic function ψ; this corresponds to smoothly switching on and off links of the network.

L, adopt explicit time-dependence and T-periodicity. We obtain

$$L(t) = \begin{pmatrix} \deg(1)(t) & -\psi\left(t+\frac{3T}{4}\right) & 0 & -\psi(t) \\ -\psi\left(t+\frac{3T}{4}\right) & \deg(2)(t) & -\psi\left(t+\frac{T}{2}\right) & 0 \\ 0 & -\psi\left(t+\frac{T}{2}\right) & \deg(3)(t) & -\psi\left(t+\frac{T}{4}\right) \\ -\psi(t) & 0 & -\psi\left(t+\frac{T}{4}\right) & \deg(4)(t) \end{pmatrix} \tag{6.13}$$

for the graph Laplacian which leads to the non-autonomous periodically forced dynamical system

$$\dot{x} = \mathcal{F}(t, x) = F(x) + P(t)H(x) \tag{6.14}$$

with time-dependent, T-periodic perturbation

$$P(t) = -L(t) \otimes \mathbb{I}_n.$$

Note that the system loses its D_4-symmetry due to the introduction of this kind of periodic forcing, but, notably, a different type of symmetry arises:

spatio-temporal symmetry. Let Σ be the matrix

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in O(4), \tag{6.15}$$

representing the rotation $(1234) \in \mathbb{Z}_4$. For the adjacency matrix we find

$$C\left(t - \frac{T}{4}\right) = \Sigma^{-1} C(t) \Sigma \quad \text{for all } t \in \mathbb{R}.$$

Similarly, we observe that

$$\operatorname{diag}(d)\left(t - \frac{T}{4}\right) = \Sigma^{-1} \operatorname{diag}(d)(t) \Sigma$$

holds for the degree matrix and, consequently, we obtain the Laplacian relation

$$L\left(t - \frac{T}{4}\right) = \Sigma^{-1} L(t) \Sigma \quad \text{for all } t \in \mathbb{R}.$$

By the properties of the Kronecker product $\otimes$ of matrices – especially by the identity $(A \otimes B)(C \otimes D) = AC \otimes BD$, which is true whenever the products AC and BD are defined – we find

$$P\left(t - \frac{T}{4}\right) = \Sigma_{(n)}^{-1} P(t) \Sigma_{(n)} \quad \text{for all } t \in \mathbb{R} \tag{6.16}$$

with $\Sigma_{(n)} = \Sigma \otimes \mathbb{I}_n \in O(4n)$ and perceive $(\Sigma_{(n)}, \frac{T}{4})$ as an element of $O(4n) \times S^1$ identifying S^1 and $\mathbb{R}/T\mathbb{Z}$. The spatio-temporal symmetry group Ξ_P of P is given by

$$\Xi_P = \left\{ (\Sigma, \theta) \in O(4n) \times S^1 \,\middle|\, (\Sigma, \theta) P(t) = \Sigma P(t - \theta) \Sigma^{-1} = P(t) \quad \forall\, t \in \mathbb{R} \right\},$$

and we figure out that

$$\Xi_P = \left\langle \left(\Sigma_{(n)}, \frac{T}{4} \right) \right\rangle \cong \mathbb{Z}_4.$$

Analogeously, the spatio-temporal symmetry group of $\mathcal{F}$ is defined as

$$\Xi_{\mathcal{F}} = \left\{ (\Sigma, \theta) \in O(4n) \times S^1 \,\middle|\, (\Sigma, \theta) P(t) = \Sigma \mathcal{F}(t - \theta) \Sigma^{-1} = \mathcal{F} \quad \forall\, t \in \mathbb{R} \right\} \tag{6.17}$$

Moreover, we find $\Xi_{\mathcal{F}} = \Xi_P$, since for $(\Sigma, \theta) \in \Xi_{\mathcal{F}}$ one has

$$\begin{aligned} \mathcal{F}(t,x) &= (\Sigma,\theta)\mathcal{F}(t,x) \\ &= \Sigma^{-1}\mathcal{F}(t+\theta, \Sigma x) \\ &= \Sigma^{-1}F(\Sigma x) + \Sigma^{-1}P(t+\theta)H(\Sigma x) \\ &= \Sigma^{-1}\Sigma F(x) + \Sigma^{-1}P(t+\theta)\Sigma H(x) \\ &= F(x) + (\Sigma,\theta)P(t)H(x). \end{aligned}$$

To be true, this identity requires $(\Sigma,\theta)P(t) = P(t)$, or, equivalently, $(\Sigma,\theta) \in \Xi_P$. Thus, the symmetries of the network perturbation $P(t)$ determine the symmetries of the whole system, while the periodic forcing of the Laplacian is responsible for the breaking of dihedral symmetries. Therefore, the relation

$$\mathcal{F}(t, \Sigma x) = \Sigma \mathcal{F}\left(t - \frac{T}{4}, x\right) \tag{6.18}$$

holds for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^{4n}$.

So, for a given D_4-symmetric autonomous coupled cell system we have introduced an external forcing making the system non-autonomous and breaking its symmetries to spatio-temporal symmetries $\mathbb{Z}_4$. In the following, we will design a switched system out of the time-dependent $\mathcal{F} : \mathbb{T} \times \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ and trace the original spatio-temporal symmetries.

The spatio-temporal symmetries $\Xi_{\mathcal{F}}$ of the system (6.14) project onto the finite subgroup $G = \langle T/4 \rangle \cong \mathbb{Z}_4$ of $S^1 = \mathbb{R}/T\mathbb{Z}$. We write $n = n(\mathcal{F}) = 4$ and discretize the interval $[0,T] \subset \mathbb{R}$ into $n_{\mathcal{F}}$ subintervals

$$I_k = \left[(k-1)\frac{T}{n_{\mathcal{F}}}, k\frac{T}{n_{\mathcal{F}}}\right]$$

of length $\sigma = \frac{T}{n_{\mathcal{F}}}$. Approximating the adjacency map $C : \mathbb{T} \to \mathbb{R}^{|\mathcal{C}|\times|\mathcal{C}|}$ by a piecewise constant map $\widehat{C}$ with

$$\widehat{C}\Big|_{I_k} \equiv C\left(\frac{2k-1}{2}\frac{T}{n_{\mathcal{F}}}\right),$$

we may perceive it as a map $\widehat{C} : \mathbb{Z} \to \mathbb{R}^{|\mathcal{C}|\times|\mathcal{C}|}$. By T-periodicity of $C(t)$, $\widehat{C}$ remains periodic in a discrete sense, more precisely

$$\widehat{C}(k + n_{\mathcal{F}}) = \widehat{C}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

With this construction, $\widehat{C}$ possesses exactly $n_{\mathcal{F}}$ values and we consider $\widehat{C}$ as a matrix family $\left\{\widehat{C}(k)\right\}_{k\in\{1,\dots,n_{\mathcal{F}}\}}$. For $k=1,\dots,4$, with (6.11), (6.12) and the T-periodicty of ψ we obtain the adjacency matrices

$$\widehat{C}(1) = C\left(\frac{T}{8}\right) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$\widehat{C}(2) = C\left(\frac{3T}{8}\right) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{C}(3) = C\left(\frac{5T}{8}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\widehat{C}(4) = C\left(\frac{7T}{8}\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

which correspond to the graphs illustrated in Fig. 6.4. Naturally, $\widehat{C}$ induces the matrix families $\widehat{\deg(d)}$, $\widehat{L}$ and $\widehat{P}$ corresponding to the degree matrix $\deg(d)(t)$, the Laplacian $L(t)$ and the network perturbation $P(t)$ and $n_{\mathcal{F}}$-periodicity is passed on to those. In this way, we end up with a discretized vector field family

$$\widehat{\mathcal{F}} = \left\{\widehat{\mathcal{F}}(k)\right\}_{k\in\{1,\dots,n_{\mathcal{F}}\}};$$

indeed, more specifically, $\widehat{\mathcal{F}}$ is a *family of coupled cell systems* in the sense of Section 6.1. Notably, none of the systems $\dot{x} = \widehat{\mathcal{F}}(k)(x)$ still has symmetry D_4, but each of them keeps $\mathbb{Z}_2$-symmetry with varying axes of symmetry (see Fig. 6.5). We aim to design a hybrid dynamical system out of $\widehat{\mathcal{F}}$ that is related to the original coupled cell systems $\mathcal{F}(x)$ and $\mathcal{F}(t,x)$ and use $\Lambda = \{1,\dots,n_{\mathcal{F}}\}$ as the vertex set of a cyclic directed graph $\mathscr{T} = (\Lambda, \mathcal{E})$ with edges

$$\mathcal{E} = \{(1,2),\dots,(n_{\mathcal{F}}-1, n_{\mathcal{F}}), (n_{\mathcal{F}}, 1)\}.$$

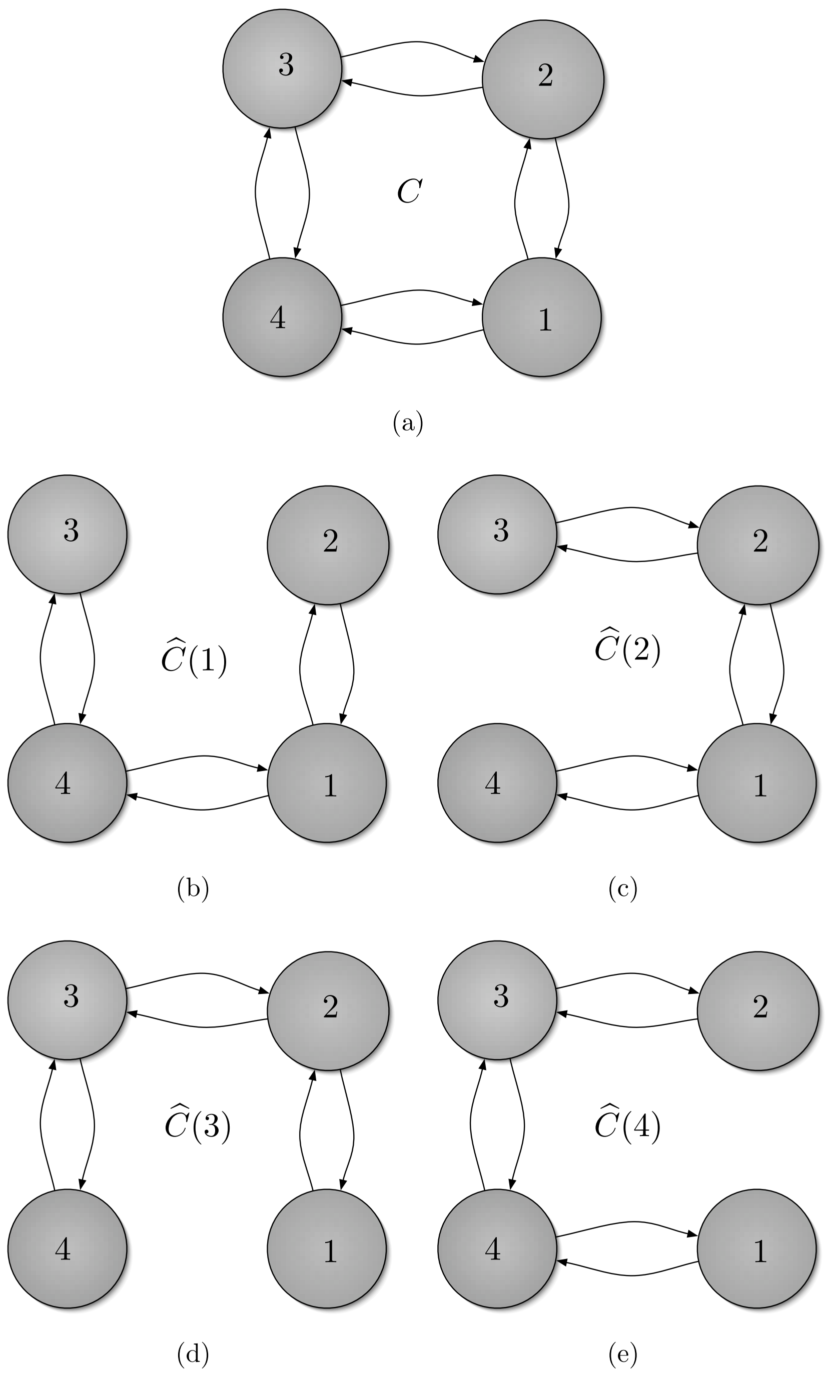


Figure 6.4: Graphs corresponding to the adjacency matrices C and $\widehat{C} = \left\{\widehat{C}(k)\right\}_{k=1,\ldots,n_{\mathcal{F}}}$

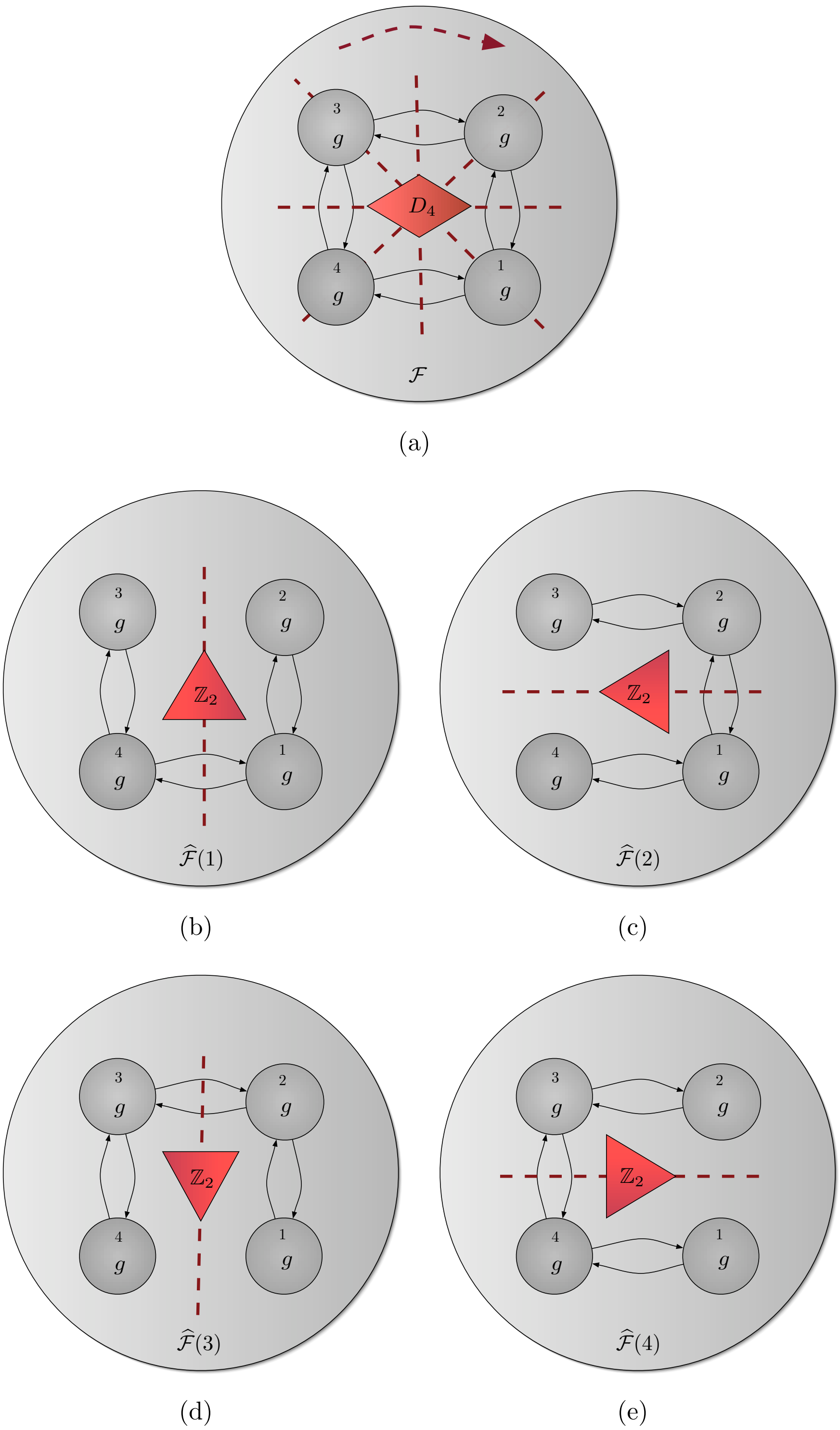


Figure 6.5: (a) Original D_4-symmetric coupled cell system $\mathcal{F}$ and (b)-(e) induced family $\widehat{\mathcal{F}}$ of coupled cell systems with symmetry $\mathbb{Z}_2$ and changing axis of symmetry

This choice is based on the temporal order determined by $\mathcal{F}(t,x)$. Thus the cyclic graph structure emulates a period of the non-autonomous system $\dot{x} = \mathcal{F}(t,x)$ in correctly directed time. The resulting switched system $\mathscr{S} = (\mathscr{T}, \mathbb{R}^{4n}, \widehat{\mathcal{F}}, \Omega)$ is sketched in Fig. 6.6. Clearly, $\mathrm{Aut}\,(\mathscr{T}) \cong \mathbb{Z}_{n_\mathcal{F}}$ and there is an isomorphism $\iota : G \cong \mathrm{Aut}(\mathscr{T})$ mapping the generator $\theta = \frac{T}{n_\mathcal{F}}$ of $G < S^1$ to the generator

$$(n_\mathcal{F}\, 1\, 2\, 3\, \ldots\, (n_\mathcal{F} - 1)) \in \mathrm{Aut}(\mathscr{T}) \leq S_{n_\mathcal{F}}.$$

With $\pi = \iota(\theta)$ Equation (6.18) transforms into

$$\widehat{\mathcal{F}}(k) \circ \Sigma = \Sigma \circ \widehat{\mathcal{F}}(\pi^{-1}(k)) \quad \text{for all } k \in \mathbb{Z}\ . \tag{6.19}$$

Hence, the switched system $\mathscr{S}$ inherits the $\mathscr{T}$-symmetries

$$\mathfrak{S} = \langle (\pi, \Sigma) \rangle \cong \left\langle \left(\Sigma, \frac{T}{n_\mathcal{F}} \right) \right\rangle = \Xi_\mathcal{F} \cong \mathbb{Z}_{n_\mathcal{F}}. \tag{6.20}$$

Now, let σ be a uniformly β-slow π^{-1}-orbital switching signal of $\mathscr{S}$. Then $\widehat{\sigma} : \mathbb{Z} \to \Lambda$ is $n_\mathcal{F}$-periodic and by Proposition 5.3.7, $((\pi, \Sigma), 1)$ is a hybrid spatio-temporal symmetry of the σ-switched system $\mathscr{S}_\sigma$. Thus, the hybrid time-$\Theta(n_\mathcal{F}\beta)$ map turns out to be of the form

$$\Phi^\sigma_{n_\mathcal{F}\beta} = \left((\pi, \Sigma)^{-1} \circ \Phi^\sigma_\beta \right)^{n_\mathcal{F}}$$

by Corollary 5.3.9.

6.4 Numerical Treatment of Orbitally Switched Systems

This section is devoted to the numerical examination of switched systems under the influence of orbital switching. To this end, we reconsider the switched system $\mathscr{S} = (\mathscr{T}, \mathbb{R}^{4n}, \widehat{\mathcal{F}}, \Omega)$ based on the cyclic switching graph $\mathscr{T} = (\Lambda, \mathcal{E})$ with $\Lambda = \{1, \ldots, n_\mathcal{F}\}$ as already discussed in the preceding section and shown in Fig. 6.6. Choosing *linear* dynamics we end up with a family of four matrices $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ that are of the form

$$\begin{aligned} A_\lambda(B, \zeta, D) &= \mathrm{diag}(B) + P_\lambda H(x; \zeta, D) \\ &= \mathrm{diag}(B) - \zeta(L_\lambda \otimes D) \qquad \in \mathbb{R}^{8\times 8}, \quad \lambda \in \Lambda, \end{aligned} \tag{6.21}$$

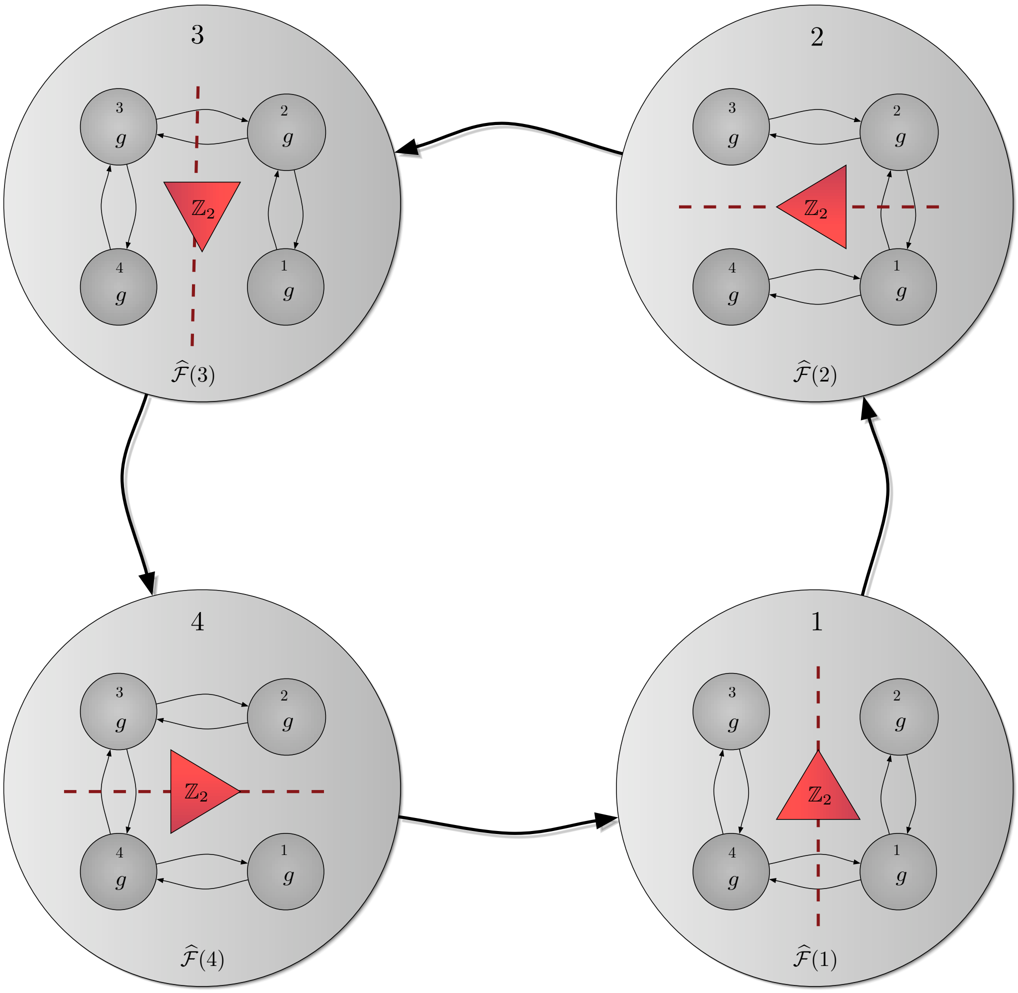


Figure 6.6: Switched system $\mathscr{S} = (\mathscr{T}, \mathbb{R}^{4n}, \widehat{\mathcal{F}}, \Omega)$ based on the family of coupled cell systems induced by the vector field family $\widehat{\mathcal{F}}$

where $B \in \mathbb{R}^{2\times 2}$ corresponds to two-dimensional linear internal dynamics and $\mathrm{diag}(B) = \mathbb{I}_4 \otimes B \in \mathbb{R}^{8\times 8}$ is the induced diagonal block matrix. Based on the adjacency matrices $C_\lambda = \widehat{C}(\lambda)$, the corresponding Laplacian matrices $L_\lambda = \widehat{L}(\lambda)$ are determined by

$$L_1 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad L_\lambda = \Sigma^{-\lambda+1} L_1 \Sigma^{\lambda-1} \tag{6.22}$$

with Σ as defined in (6.15). For our numerical investigations, we fix

$$B = \begin{pmatrix} -0.39 & -0.4 \\ 0.04 & -0.39 \end{pmatrix}, \quad D = \mathbb{I}_2 \quad \text{and} \quad \zeta = -0.111\ . \tag{6.23}$$

Additionally, for reasons of comparison, we consider the unforced system (6.8) with complete D_4-symmetry yielding the matrix

$$A_0 = \mathrm{diag}(B) - \zeta L \otimes \mathbb{I}_2$$

with

$$A_0 = \begin{pmatrix} -0.168 & -0.4 & -0.111 & 0 & 0 & 0 & -0.111 & 0 \\ 0.04 & -0.168 & 0 & -0.111 & 0 & 0 & 0 & -0.111 \\ -0.111 & 0 & -0.168 & -0.4 & -0.111 & 0 & 0 & 0 \\ 0 & -0.111 & 0.04 & -0.168 & 0 & -0.111 & 0 & 0 \\ 0 & 0 & -0.111 & 0 & -0.168 & -0.4 & -0.111 & 0 \\ 0 & 0 & 0 & -0.111 & 0.04 & -0.168 & 0 & -0.111 \\ -0.111 & 0 & 0 & 0 & -0.111 & 0 & -0.168 & -0.4 \\ 0 & -0.111 & 0 & 0 & 0 & -0.111 & 0.04 & -0.168 \end{pmatrix}.$$

Now, we detect instability of A_0 caused by a conjugated pair $0.054 \pm 0.1265i$ of eigenvalues. By (6.20), the switched linear system $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ has a cyclic hybrid symmetry group $\Xi_{\mathscr{S}} = \langle(\Sigma, \pi)\rangle$ and we consider π-orbital switching σ_π generated by $\pi = (1432) = (1234)^{-1} \in \mathbb{Z}_4$. Note that $\mathrm{ord}(\pi) = 4$ and that $\Gamma = \Gamma_\pi$ acts transitively on Λ.

Contrary to A_0, the matrices of $\mathcal{A}$ turn out to be stable throughout, since $\mathrm{Re}(\xi) < 0$ for every eigenvalue ξ of A_λ, $\lambda \in \Lambda$. The numerical simulation is organized as follows: We start out from the slightly perturbed equilibrium in the origin $0 \in \mathbb{R}^8$, driven by the unstable dynamics of the unforced completely D_4-symmetric system represented by A_0, before we introduce π-orbital switching as described above and, finally, return to the unswitched system to observe the system's unperturbed behavior again putting a stress on the effects of orbital switching. Figure 6.7 demonstrates the stabilizing effect of orbital switching with speed $\varsigma = 24$; after stopping the switching process, the unstable dynamics force the system to drift away from the origin again.

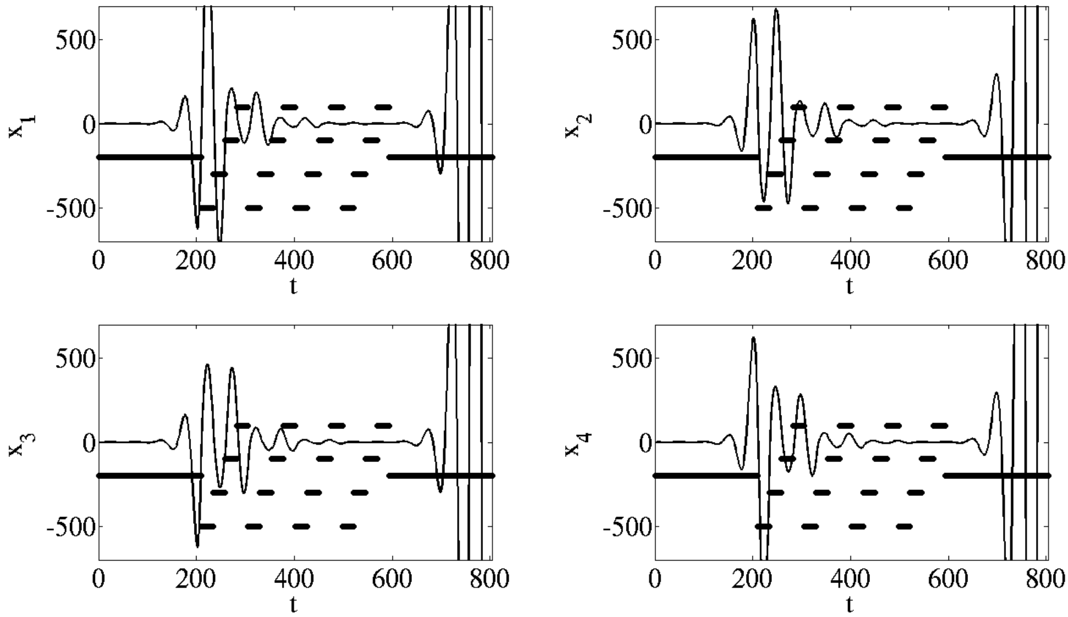


Figure 6.7: Stabilization by π-orbital switching with $\varsigma = 24$. The first coordinate of each cell is plotted separately. The bold line segments document the switching history of the system: On the time intervals $[0, 210]$ and $[594, 804]$, there is no switching, and the dynamics are ruled by A_0. In between, π-orbital switching takes place which guides the system back to the origin.

In order to simplify the numerical treatment of the switched system in question, we consider its discrete time version $x(k+1) = \widetilde{A}_{\sigma_\pi(k)} x(k)$ with orbital switching signal $\sigma_\pi : \mathbb{Z} \to \Lambda$. Let $\|\cdot\|$ denote the spectral norm of matrices which is given by

$$\|A\| = \sqrt{\varrho(A^T A)} = \sigma_{\max}(A), \tag{6.24}$$

where $\sigma_{\max}(A)$ denotes the largest singular value of A. Since $\exp(At) = \exp(A)^t$ for a matrix $A \in \mathbb{R}^{n \times n}$ and $t \in \mathbb{Z}$, along the lines of the proof of

Theorem 5.4.3, we obtain the following similar result for discrete time which is published in the article [HPD11] by Hage-Packhäuser and Dellnitz.

6.4.1 Theorem. *Let $\mathscr{S} = (\mathscr{T}, \mathcal{D}, \mathcal{A}, \Omega)$ be a discrete-time switched linear system with $\mathscr{T}$-symmetry group $\mathfrak{S}$. Let $(\pi, \Sigma) \in \mathfrak{S}$ be a $\mathscr{T}$-symmetry and $\lambda_0 \in \Lambda$ such that e_0^π and $e_0^{\pi^{-1}}$ are edges of $\mathscr{T}$. Assume that there exist $\lambda \in \Gamma_\pi \lambda_0$, $\beta \in \mathbb{N}$ and $\rho_\lambda < 1$ such that $\left\|\exp(A_\lambda)^\beta\right\| \leq \rho_\lambda$. Then, if Γ_π is normal in Γ, the switched system $\mathscr{S}$ is asymptotically stable under uniformly β-slow π^ν-orbital switching for all $\nu \in \Gamma$.*

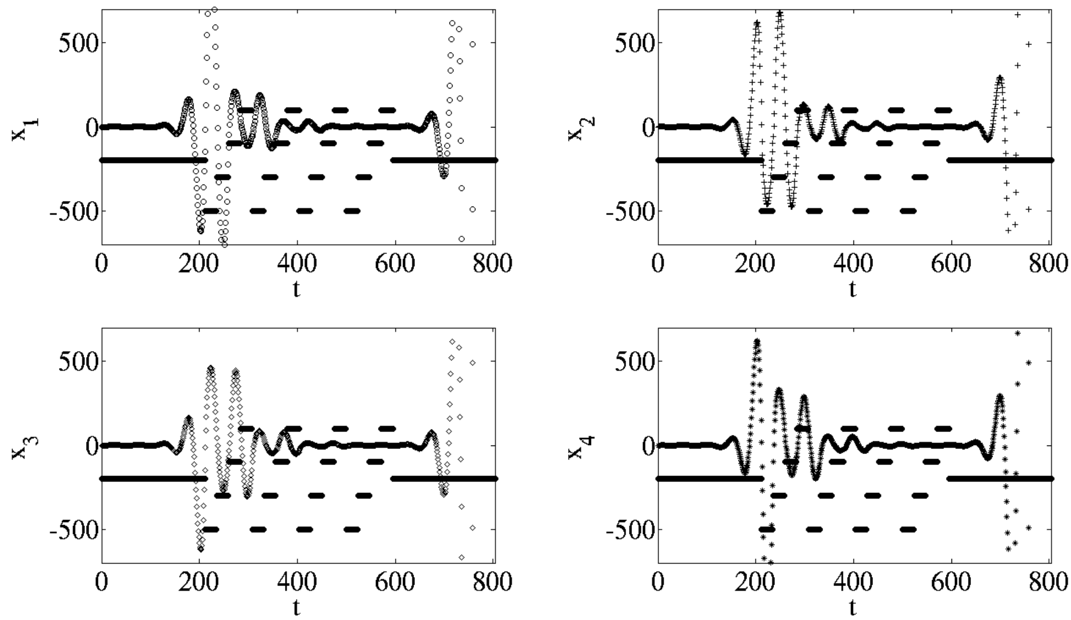


Figure 6.8: Asymptotic stability via 24-slow π-orbital switching of the discrete-time switched linear system $x(k+1) = \widetilde{A}_{\sigma_\pi(k)} x(k)$.

We thus turn to the induced discrete-time orbitally switched system $x(k+1) = \widetilde{A}_{\sigma_\pi(k)} x(k)$, where we recall the notation $\widetilde{A}_\lambda = \exp(A_\lambda)$. For A_1, we find that $\beta = 24$ is the minimal non-negative integer satisfying

$$\left\|\widetilde{A}_1^\beta\right\| \leq \eta_1 < 1 \quad \text{with} \quad \eta_1 = 0.8915.$$

Thus, by means of Theorem 6.4.1, the discrete-time switched system in question is asymptotically stable for 24-slow π-orbital switching. In Figures 6.8

and 6.9, we visualize stabilizing 24-slow orbital switching of the discrete-time system. Since $\Gamma = \Gamma_\pi$ holds, Γ_π is normal in Γ and Theorem 6.4.1

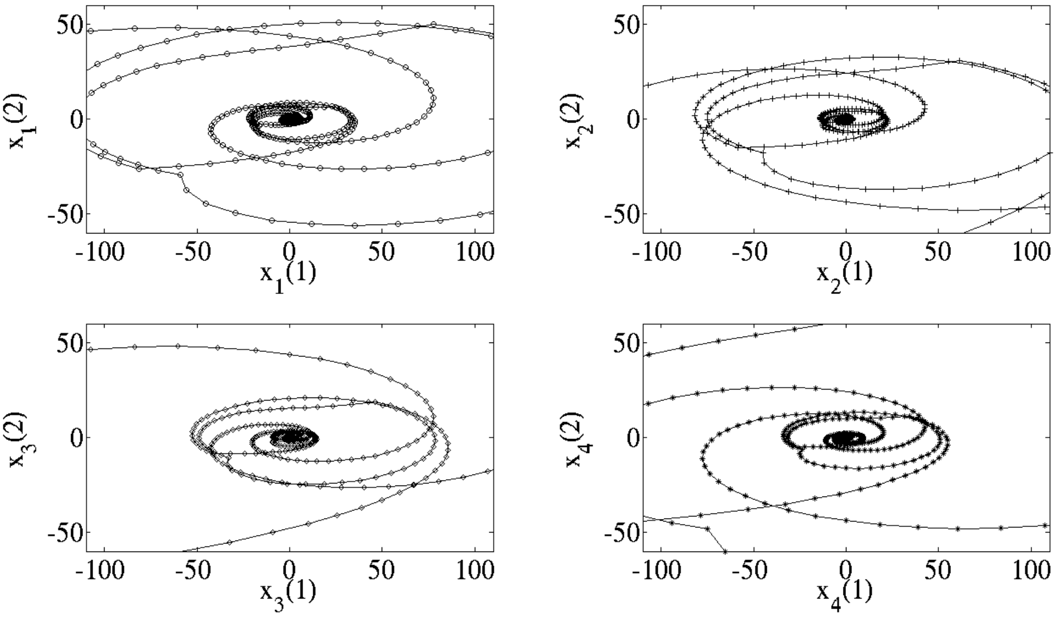


Figure 6.9: Phase plots of the subsystems visualizing asymptotic stability via 24-slow π-orbital switching.

predicts asymptotic stability under 24-slow $\gamma^{-1}\pi\gamma$-orbital switching for all $\gamma \in \Gamma = \langle(1234)\rangle = \mathbb{Z}_4$. In this case, we solely have $\gamma^{-1}\pi\gamma = \pi$, due to commutativity of Γ.

Finally, after having laid the foundations for the treatment of temporally varying dynamical system networks, we discuss a more interesting example building on the preceding one (see Figure 6.6). Again based on the D_4-symmetric network displayed in Figure 6.2, we consider the more intricate class of switched systems illustrated in Figure 6.10 which is determined by the matrices $A_\lambda = A_\lambda(B, \zeta, D)$ as in (6.21) with Laplacians L_λ as in (6.22) for $\lambda = 1, 2, 3, 4$ and

$$L_5 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad L_6 = \Sigma^{-1} L_5 \Sigma.$$

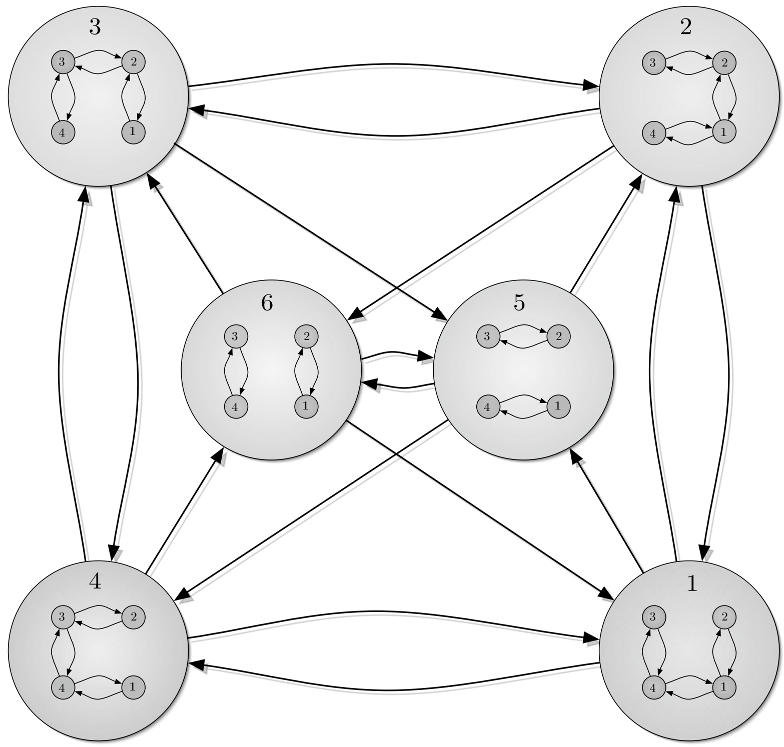


FIGURE 6.10: Structure of a switched system $\mathscr{S}$ admitting the hybrid symmetry group $\Xi_{\mathscr{S}} \cong \mathbb{Z}_4$ generated by (Σ, π) with $\pi = (1234)(56)$. The intransitive action of $\Xi_{\mathscr{S}}$ on $\Lambda = \{1, 2, 3, 4, 5, 6\}$ causes two fundamentally distinct orbital switching strategies which may lead to different stability results.

Let π denote the permutation $(1234)(56) \in S_6$. Again, the switched system exhibits cyclic symmetry given by the hybrid symmetry group $\Xi_{\mathscr{S}} = \langle(\Sigma, \pi)\rangle \cong \mathbb{Z}_4$ with Σ as in (6.15). Note that – in this case – the discrete part $\Gamma = \Gamma_\pi = \langle\pi\rangle$ does not act transitively on the discrete states Λ since we detect the existence of two orbits $\{1, 2, 3, 4\}$ and $\{5, 6\}$. For our computations, we let B and D be as above (cf. (6.23)), but use $\zeta = -0.408$. At first, we consider π-orbital switching $\sigma_\pi^{\lambda_0}$ with initial state $\lambda_0 = 5$ which is admissible since $e_0^\pi = (5, 6) \in \mathcal{E}$. For the system matrix A_5, we find

$$\left\|\widetilde{A}_5\right\| \leq \eta_5 \quad \text{with} \quad \eta_5 = 0.6596 < 1.$$

Thus, by Theorem 6.4.1, the discrete-time system is asymptotically stable under (1-slow) orbital switching σ_π^5. However, when we consider π-orbital switching with $\lambda_0 \in \{1, 2, 3, 4\}$ which is admissible as well, then we observe that the system does not stabilize: It turns out that the corresponding system matrices have eigenvalues outside the unit disc keeping them away from Schur stability.

Chapter 7

Conclusion

This final chapter encompasses a rough summary of the thesis' findings as well as an outlook on possible future research directions concerning hybrid symmetries.

The Essence of This Thesis

The general aim of this thesis has been the analysis of time-varying equivariant dynamical systems with changing symmetry properties from a hybrid point of view with a particular consideration of symmetries. As a prominent prototypical example which the world is laced with when trying to grasp it mathematically, the class of time-varying dynamical system networks (or coupled cell systems) is permanently present. Its symmetries naturally stem from the underlying coupling architecture and its explicit time-dependence is assumed to originate from a temporal evolution of the coupling topology. Such systems are relevant for modelling and analyzing mobile communication networks or multi-agent systems, for instance. From a more abstract point of view, coupled cell systems with temporally varying coupling network may be interpreted in terms of network perturbations or link failures.

First of all, *hybrid automata* and *switched systems* – which intuitively seem to be close to each other at first sight, but turn out to be distinguished by conceptual dissimilarities – have been discussed in a non-standard unifying manner fathoming their tense relationship. The latter is resolved by the incorporation of a distinguished subclass of switched systems into the hybrid automata framework.

Motivated by the question for the overall symmetry information of a non-autonomous dynamical system with time-dependent symmetry properties, a global symmetry concept for hybrid dynamical systems is unfolded broad in scope. More precisely, this symmetry framework is developed for hybrid automata composed of classically equivariant dynamical systems. The detailed accomplishment of this construction which is designed to act as a hybrid analog to classical dynamical system symmetries leads to the notion of *hybrid symmetries* translating to a weak form of equivariance for structured vector field families opposed to the classical equivariance of a single vector field. This *weak equivariance* may be interpreted as a kind of *spatio-spatial symmetry* when compared to traditional spatio-temporal symmetries. Algebraic properties as well as the immediate consequences of hybrid symmetries for the dynamics of hybrid automata are studied uncovering a great structural similarity between hybrid and classical symmetries. However, in the face of fixed-point spaces the two concepts drift apart and unveil their diversity since hybrid fixed-point spaces are shown to be not naturally invariant.

Built on the notion of hybrid symmetries, the concept of *hybrid spatio-temporal symmetries* is developed for periodic executions. These are utilized to decompose certain return maps of hybrid automata with prescribed switching strategies giving a fixed-point formulation of spatio-temporally symmetric executions with respect to specific symmetry-related maps.

Against this theoretical background, *switching signals* are spotlighted and a distinguished class of switching signals – *orbital switching* signals – is put into the center of interest. This way of switching is generated by hybrid symmetries and it is proven to give rise to hybrid spatio-temporal symmetries with respect to the induced switched system. In this context, the temporal aspect of switching is discussed and figured out to essentially shape the symmetry properties of the induced system.

After examining the influence of conjugation on orbital switching signals, we succeed in providing sufficient conditions concerning the *stabilization* of switched linear systems for conjugacy classes of orbital switching signals as a consequence of the symmetry-based return map decomposition – all with a view to the temporal composition of the signals.

While the considerations and results reported so far are general in the sense that they hold for any hybrid dynamical system composed of arbitrary equivariant dynamical systems or switched system, respectively, the final part of the thesis is taken up by the treatment of *time-varying dynamical system networks* in the light of the afore developed theory. Having approached the issue of dynamical system networks broadly in terms of coupled cell systems, we restrict the attention to a special class of globally symmetric coupled systems and derive a switched system from the discretization of an external periodic forcing of the coupling network. We succeed in detecting non-trivial hybrid symmetries and both analytically and numerically consider orbital switching strategies.

This brings us full circle insofar as the varying dynamical system networks considered at the beginning are identified as orbitally switched systems induced by hybrid symmetries and are thus embedded into the theory of symmetric hybrid automata set up by means of this thesis.

From this retrospect, we expose the following findings and conclusions to be drawn from this work:

- Coupled cell systems whose underlying coupling network is subject to instantaneous temporal changes fall into the category of hybrid systems and may be described in terms of hybrid automata or switched systems.
- Hybrid automata may be considered to be switched systems together with a special class of switching signals. Contrariwise, if restricted to switching signals which exhibit a special structure, switched systems give rise to structurally similar hybrid automata.
- There is a structurally similar generalization of the classical dynamical system symmetry concept for hybrid dynamical systems: The classical equivariance of a vector field is replaced by a weak equivariance property for a structured vector field family and the collection of hybrid symmetries exhibits the algebraic structure of a group acting on the set of executions, i. e. hybrid symmetries preserve the dynamics.
- The hybrid analog of fixed-point spaces reveals considerably weaker invariance properties than expected from the classical theory.

- Hybrid spatio-temporal symmetries of periodic executions and of switched systems themselves are a natural generalization of hybrid symmetries and can be utilized to decompose the return maps with respect to periodic switching signals giving an approach to stabilization issues.
- Orbital switching signals adequately describe cyclically moving network perturbations in certain time-varying coupled cell systems. From a superior point of view, orbital switching can be considered as an implementation of a hybrid system's genetical information and may thus be classified as an instance of self-organized switching opposed to externally computed signals.
- In general, given a switched system possessing non-trivial hybrid symmetries and an arbitrary switching signal, symmetries hardly survive in the induced switched system. Orbital switching constitutes a prime example for keeping symmetries, since orbital switching gives rise to hybrid spatio-temporal symmetries of the induced system and thus clears the way for stability analysis.

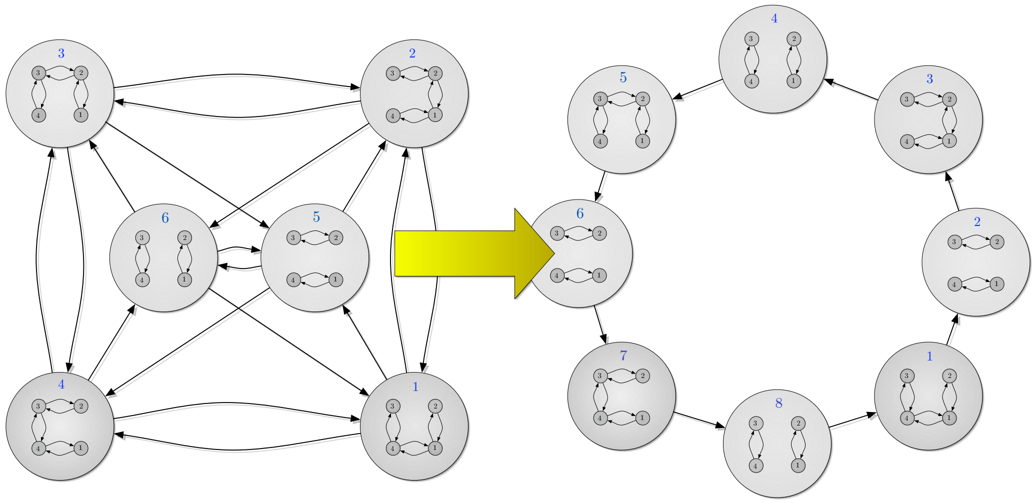
Outlook and Directions of Further Research

Undoubtedly, in a work like this where an established concept trailing a mature theory is implanted into another field, there are lots of directions which could be of interest for future research projects. However, I feel obliged to make a reasonable choice to present here and restrict myself to three broad and differently directed, but important visions of possible future work.

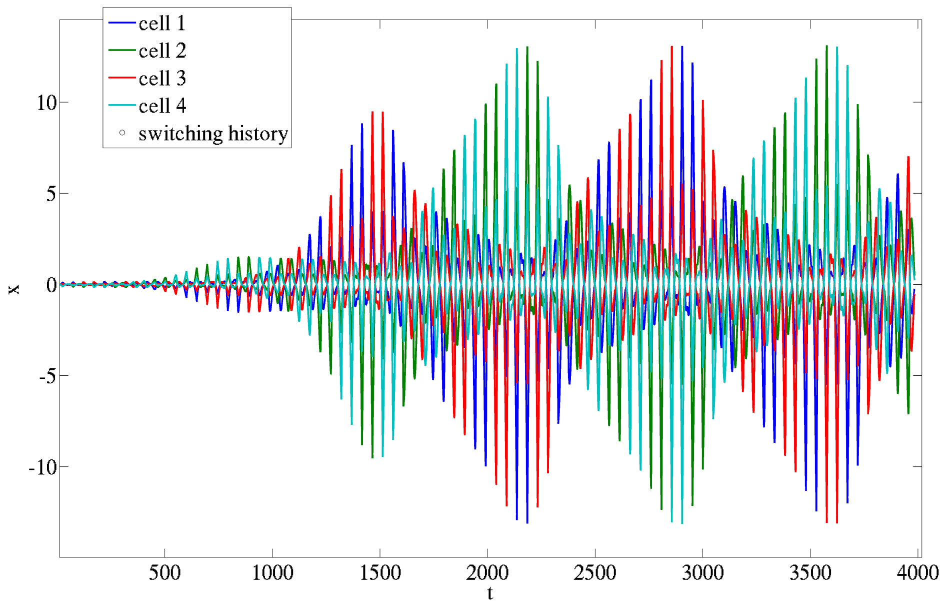
Weaker Forms of Orbital Switching

The specific switching strategies developed and analyzed in this thesis are based on symmetry properties of the hybrid system under examination. More precisely, an orbital switching signal is characterized by iteratively running through a cyclic group orbit. Appearingly, this is the strictest and most regular way of switching on the grounds of symmetry. Its advantage clearly lies in its compact formulation as a map $\widehat{\sigma} : \mathbb{Z} \to \Lambda$ with $\widehat{\sigma}(k) = \pi^{-k}(\lambda_0)$ and its iterative character.

Certainly, there are various different possibilities of switching making use of hybrid symmetry information. Figure 7.1 provides a numerical glimpse of hybrid dynamics driven by symmetry-related switching which is not orbital. While it is natural in some sense to stay in a fixed group orbit while switching, it appears to be of immense interest to switch *between* group orbits. However, it remains unclear how to organize and algebraically arrange the switching in this far more general case.



(a)



(b)

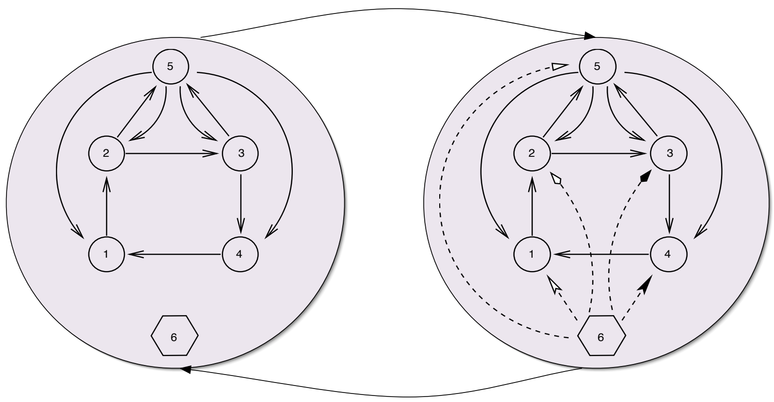
Figure 7.1: (a) Switched System with non-trivial symmetries and a related periodic switching signal described by its induced discrete trace; (b) Sample execution with respect to the above switching signal.

Local Hybrid Symmetries

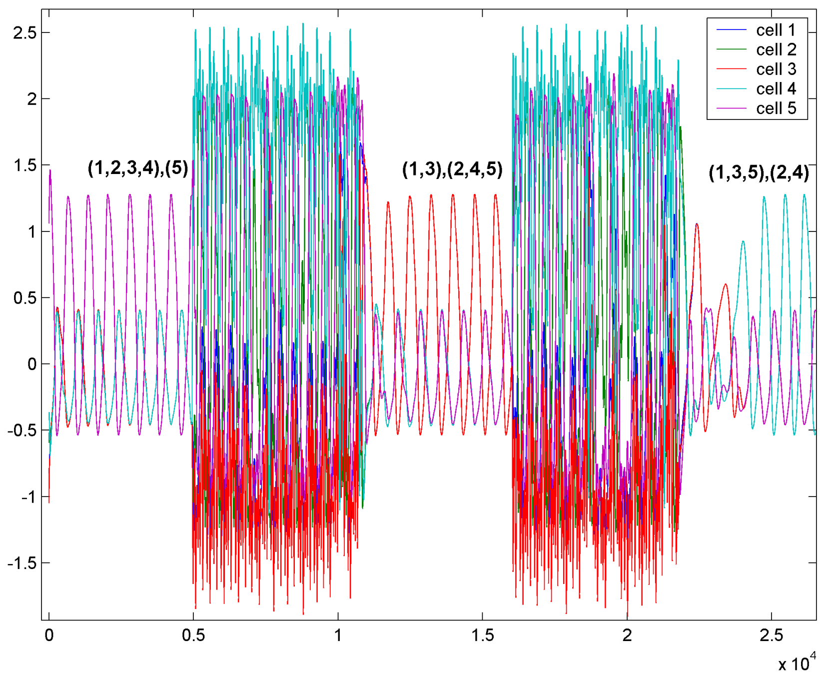
The following direction aims at a refinement of the hybrid symmetry concept and thus at a deeper structural understanding of hybrid dynamical systems in terms of symmetries.

In this thesis, *global* symmetry properties of hybrid dynamical systems have been treated. As indicated in connection with coupled cell systems, global symmetries are of a highly restrictive nature and extremely sensitive with respect to structural perturbations – in contrast to *local* symmetries. However, the drawback of such local objects principally lies in the fact that in general they cannot manage to give rise to global structures such as groups, for instance, if algebraic properties are considered.

If one shifts the view from coupled cell systems to hybrid systems, it stands to reason to establish the notion of *local hybrid symmetries* for hybrid automata by combining local symmetry information of the transition graph with local symmetries of the single dynamical systems the hybrid system is composed of. This is of particular interest for the study of symmetry-induced synchrony patterns and their temporal development. Figures 7.2 and 7.3 visualize first computational experiments and hint at the form of hybrid dynamics occuring in the setting of coupled cell systems without global symmetries.

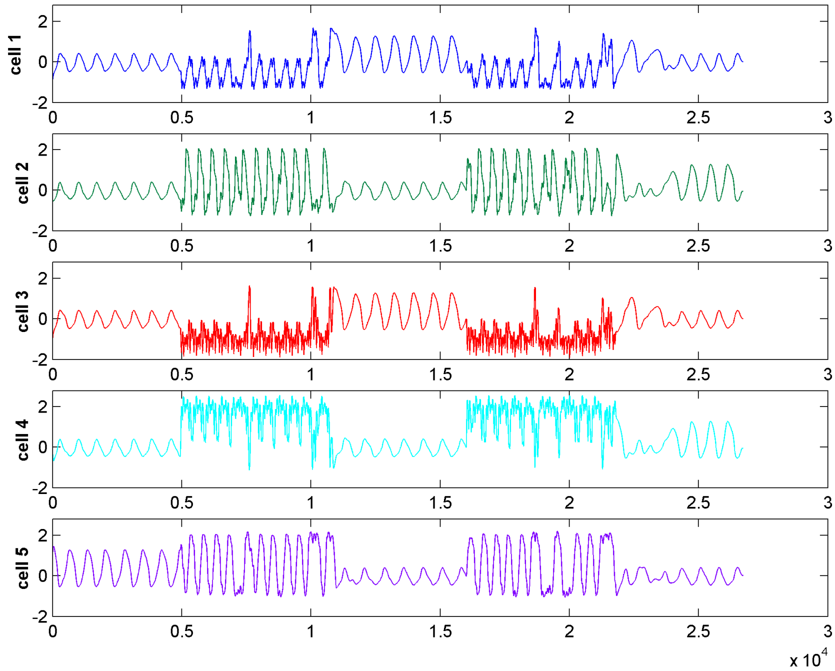


(a)

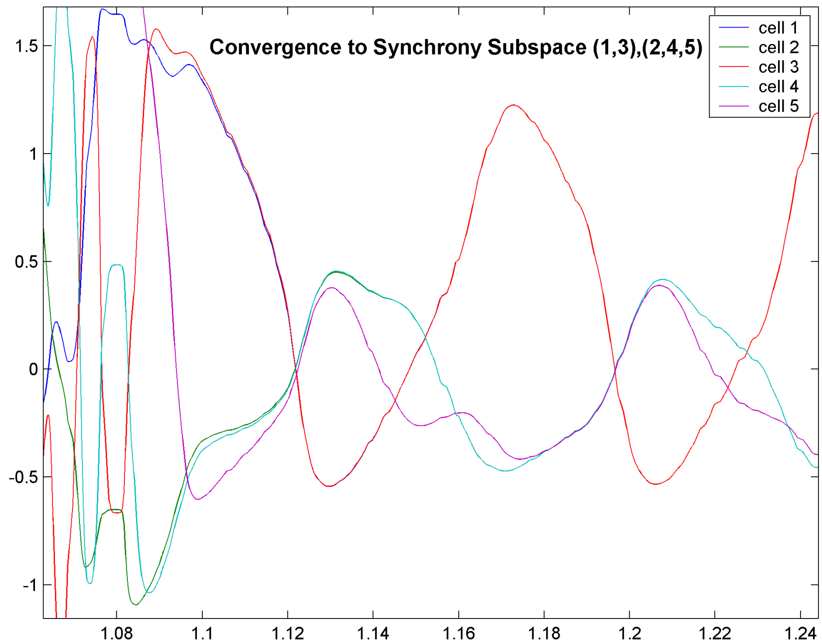


(b)

Figure 7.2: (a) Switched system composed of two coupled cell systems without global symmetries. (b) Hybrid dynamics showing a series of synchrony patterns among the cells.



(a)



(b)

FIGURE 7.3: (a) Projections of the execution displayed in Fig. 7.2 (b) to the single cell phase spaces emphasizing synchrony relations. (b) Detail enlargement of Fig. 7.2 (b).

Bifurcation Theory in Presence of Hybrid Symmetries

This direction of further research is qualitatively different from the ones before; it is concerned with the understanding of deeper effects of present symmetries.

Typically, dynamical systems derived from real world phenomena and analyzed for the sake of comprehending such aspects of the real world depend on many different types of parameters besides time. What is more, these parameters are usually not temporally constant themselves. Since it is a common observation that dynamics may change qualitatively under varying parameters, parameter variation is an integral part of understanding various phenomena occuring in many kinds of dynamics. The prediction of possible changes in the dynamics under parameter variation is the task of *bifurcation theory*.

In the case of time-varying dynamical system networks modelled as switched systems, there are at least two natural parameters that deserve special attention: the first one is the *coupling strength* occuring in coupled systems which controls the manner systems are coupled to each other; the second parameter of interest is the *switching time* or more generally the temporal composition of switching signals. With regard to hybrid symmetries, an *equivariant bifurcation theory* for hybrid systems needs to be developed. The contents of this thesis lay the foundations for a theory like that by providing a symmetry notion for hybrid automata and a general treatment of their consequences on the dynamics.

In the context of bifurcations, the notion of *stability* is of major importance since stabiliy may be lost or gained via bifurcation. This directly leads to the necessity of understanding well the relation of switching in hybrid systems and the connected stability properties. As exemplarily discussed in Chapter 6, the stability properties of a system can change due to the forced breaking of symmetries; in the example, classical (and trivially also hybrid) D_4-symmetry of an unstable system is broken to hybrid cyclic $\mathbb{Z}_4$-symmetry accompanied by the effect of stability in combination with an appropriate way of switching. In a way, this can be interpreted as the starting point of an extensive treatment of *stabilization by symmetry-breaking* in the context of hybrid systems.

What gives this example a special taste of its own, is the fact that the algebraic structure of the *original* system and the *spatial average* of the switched system practically coincide, i. e. while stability properties shift, the algebraic information carried by symmetries remains on average –

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